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DERIVATIVE OF EQUATIONS

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Abstract : This paper was based on sample diagnostic questions for the concept of derivatives with the aim of improving students' ability to correctly apply the rules for finding derivatives of equations. The study has confirmed that the derivative is one of the concepts students have difficulty, as indicated in the literature. In this paper we can define the exact or instantaneous rate of change of a function.

Keywords : Derivatives, Equations, Concept, Difficulty, Functions.

I. Rate of Change

The derivative of a function at a chosen input value describes the best linear approximation of the function near that input value. For a real valued function of a single real variable, the derivative at a point equals the slope of the tangent line to the graph of the function at that point. In higher dimensions, the derivative of a function at a point is a linear transformation called the linearization. The process of finding a derivative is called differentiation.

The fundamental theorem of calculus states that differentiation is the reverse process to integration. The mathematical relationship between an independent and dependent dimension. The relationship expressed in the form of a mathematical function defines a situation for a given set of conditions and properties. Since we allowed the independent dimension to change, it was no longer a fixed quantity but became a variable.

The graph of the function reflected a horizontal change of the independent variable with a vertical change in the dependent dimension. This graph allows us to visualize how a situation changes with respect to a change in the conditions. To find out how much the function, $f\left(x\right)$, changes between two points x_1 and x_2 we simply enter in the two values for the independent variable *x* and then calculate the difference between the dependent variable, *f*, for those given conditions.

Remember that a variable is nothing more than a dimension that is allowed to change or take on any value. Thus, from x_1 to x_2 the change in the independent variable, referred to as Δx is :

$$\Delta x = x_1 - x_2$$

The corresponding change in the dependent variable, referred to as Δf is :

$\Delta f = f\left(x_2\right) - f\left(x_1\right)$ Δx refers to an interval over which we are analyzing the change in the dependent dimension, *f*.... The second point x_2 can be written in terms of the first point x_1 plus the change in the variable Δx .

$$x_2 - x_1 = \Delta x$$

$$\therefore x_2 = x_1 + \Delta x$$

Therefore the change in the dependent variable over an interval $x_2 x_1$ from $\Delta f = f(x_2) - f(x_1)$ which can also be written as :

$$\Delta f = f(x_1 + \Delta x) - f(x_1)$$

where $\Delta x = x_2 - x_1$

For example the change in the function $f(x) = 3.x^3 x = 3$ to $x = 5$, where

Δx is $5 - 3 = 2$ is :

$$\Delta f = f(x_1 + \Delta x) - f(x_1)$$

$$\Delta f = f(3+2) - f(3)$$

$$\Delta f = 3.5^3 - 3.3^3 = 375 - 81$$

$$\Delta f = 294$$

II. Average Rate of Change

We have learned that a change in the independent variable is defined as $\Delta x = x_2 - x_1$, and the corresponding change in the dependent variable over this interval is $\Delta f = f(x_1 + \Delta x) - f(x_1)$. The question we now must ask ourselves is how can we measure the relative change of the dependent variable with respect to the independent variable? In other words how can we calculate how much more or less Δf changed compared to Δx.

To calculate how much more $f(x)$ changed over an interval from x_1, x_2, we simply divide the change in *f* over the change in *x* for the interval. Thus we divide, Δf by the interval over we which are evaluating it, Δx which is equal to $x_2 - x_1$. Thus the relative change of *f* with respect to *x* over an interval Δx is defined as :

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

While this expression may seem rather simple, it does require some explanation, By dividing the change in *f* by the change in *x* what we are doing is calculating how much more *f* changed for a given change in *x*. For example in the function, $f(x) = 3.x^3$, when *x* changed from 3 to 5, *f* changed from 81 to 375. Over this interval of from *x* = 3 to *x* = 5, the Δf was 294.

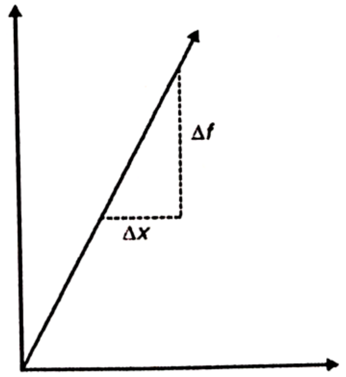
Thus the relative change in *f* with respect to a change in the independent variable *x* is:

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The value of 147 tells us that *f* changes 147 times faster than *x* over that interval of Δx from *x* = 3 to *x* = 5 only. Thus for each unit change in *x*, $\Delta x = 1$, the corresponding change in *f* is 147. We can therefore define the rate of change of a function with respect to its independent variable to be :

$$\frac{\Delta f}{\Delta x} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

The value, called the rate of change of the function, refers to how much more of less $f(x)$ changes for a unit change in *x*. It is only valid is for the interval under consideration, $\Delta x = x_2 - x_1$.



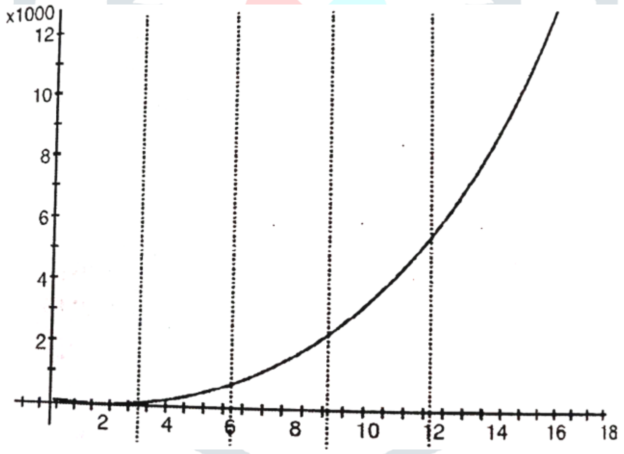
Another way of understanding what rate of change of a function means is to look at the steepness of the line connecting the two end points of the interval under consideration.

As Δf increases, the steepness of the line connecting the two endpoints will increase. Thus, the greater the rate of change of the function, the greater its slope or steepness over the interval under consideration.

Since slope and rate of change are synonymous, then how is rate of change defined for functions whose graphs do not have constant slopes? For example, from x = 9 to x = 12 of $f\left(x\right)=3.x^3$, the change in f is 2997. Thus the rate of change of the function over the interval is :

$$\frac{\Delta f}{\Delta x}=\frac{2997}{3}=999$$

This value is significantly higher than the rate of change calculated for the previous interval from x = 3 to x = 5. We can only conclude that the rate of change or slope of the graph must be increasing and is not constant over an interval $\Delta x=x_2-x_1$. Look at the graph of the function $f\left(x\right)=3.x^3$ to understand how this might be so ;



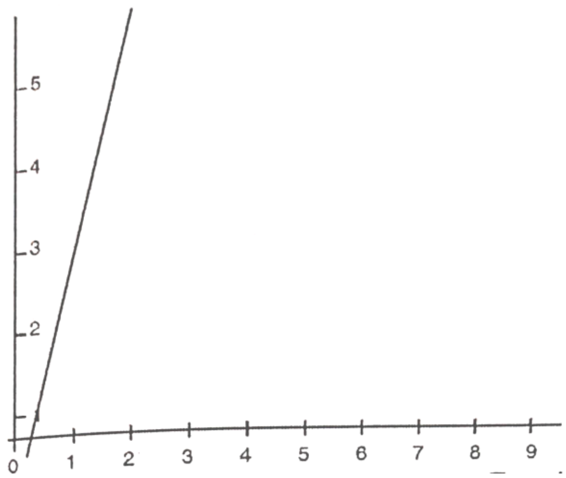
Since the rate of change of a function can change, then we have to come up with a more refined definition of rate of change. We can define the average rate of change of a function over an interval $\Delta x=x_2-x_1$ to be equal to

Average rate of change = $\dfrac{\Delta f}{\Delta x}=\dfrac{f\left(x_1+\Delta x\right)-f\left(x_1\right)}{\Delta x}$

In the next section we will take a closer look at how we can define the exact or instantaneous rate of change of a function.

III. Instantaneous Rate of Change

In this section we will take a much closer look at rate of change and see how we can define the instantaneous rate of change of a function at any point or value of the independent variable.



Let us begin with a study of the simple $f(x)=3x$. Functions of the form $f(x)=cx$, where *c* is a constant, express direct relationships. This is because the value of the function, *f*, is a constant multiple or fraction of the independent variable.

Thus *f* is said to be directly proportional to *x*. The graph of the function $f(x)=3x$ looks like. From *x* = 1 to *x* = 3, $\Delta x=2$ and $\Delta f=9-3=6$. Thus, over this interval of $\Delta x=2$, Δf equals 6. The average rate at which *f* changed with respect to *x* is by definition, $\frac{\Delta f}{\Delta x}=\frac{6}{2}=3$. For each unit change in *x*, the change in *f* is 3. This tells us that *f* is changing three times faster that *x* is changing over the interval from *x* = 1 to *x* = 3. We can now look at the interval from *x* = 2 to *x* = 4 where Δx equals 2.

$$\Delta f=f(x_1+\Delta x)-f(x_1)$$

$$\Delta f=f(2+2)-f(2)$$

$$\Delta f=3.(4)-3.(2)$$

$$\Delta f=6$$

Thus

$$\frac{\Delta f}{\Delta x}=\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}=\frac{6}{2}=3$$

Over this interval, the rate of change is the same constant, 3. This leads us to conclude that the rate of change of the function over any interval is a constant, 3. This can be proven by the definition of rate of change:

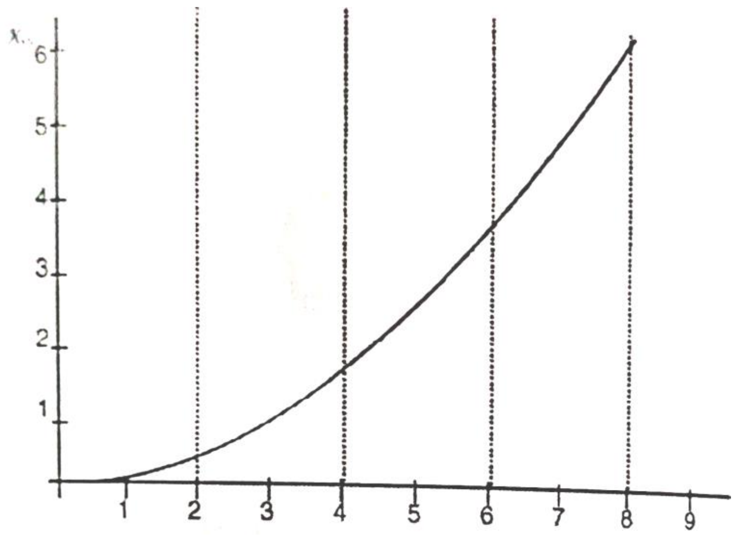
$$f(x)=3x$$

$$f(x+\Delta x)=3(x+\Delta x)$$

$$\frac{\Delta f}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{3x+3\Delta x-3x}{\Delta x}$$

$$\frac{\Delta f}{\Delta x}=\frac{3\Delta x}{\Delta x}=3$$

The rate of change of the direct function $f(x)=cx$ is *c* and *x* is constant over an interval, Dx. The graph of a function $f(x)=cx$ is therefore an increasing straight line with a constant slope or steepness equal to *c*. For this reason functions of the form $f(x)=cx$ are also called linear functions since their graphs are straight lines with a constant rate of change.



On the other hand, the graph of the function $f(x) = x^2$ is not a straight line. Unlike a line, the rate of change of $f(x)$ is not the same constant over any interval Dx. To define the rate of change for the function we will have to derive a more precise way of defining rate of change of function. To do this we will analyze the function over small intervals of Dx.

The first Point to notice is that the rate of change of the function varies with *x*. When *x* is small, *f* does not change that much as compared to how much it changes when *x* is large. We can conclude that the rate of change of *f* with respect to *x*, is not constant over any interval, Dx, but varies with *x*. To begin our analysis let us divide up the graph into intervals of $Dx = 2$ and study what is happening in each of these intervals separately. From *x* = 0 to *x* = 2, *Dx* = 2, the change in *f*, *Df* is $f(2) - f(0) = 4$. From *x* = 6 to *x* = 8, *Dx* = 2, the change in *f*, *Df* is $f(8) - f(6) = 28$.

Clearly, as *x* increases the rate of change of the function is increasing and is not a constant as in our study of the line where the function changed at the same rate as *x* increased.

Returning back to the graph of $f(x) = x^2$; we calculated that from *x* = 0 to *x* = 2 the change in *f* was 4. We can then conclude that from *x* = 0 to *x* = 2 the average rate of change of the function over that particular interval is 2 or :

$$\frac{\Delta f}{\Delta x} = \frac{4}{2} = 2$$

Remember it is called average because this rate of change is only valid from *x* = 0 to *x* = 2. Now let us consider the interval from *x* = 2 to *x* = 4 where once again the change in *x*, *Dx* is 2. The change in *f* of the graph is equal to :

$$\begin{aligned}\Delta f &= f(x+\Delta x) - f(x)\\ &= f(2+2) - f(2)\\ &= f(4) - f(2)\\ &= 16 - 4\\ &= 12\end{aligned}$$

Thus the average rate of change of the function over this interval is equal to

$$\frac{\Delta f}{\Delta x} = \frac{12}{2} = 6$$

This value is greater than the value we observed from *x* = 0 to *x* = 2. This implies that from *x* = 2 to *x* = 4, $f(x)$ is increasing at a greater rate than from *x* = 0 to *x* = 2. This is despite the fact that in both cases the *Dx* = 2. The rate of change is therefore not constant over but is increasing with *x*. When we say the rate of change of $f(x)$ from *x*

= 2 to x = 4 is 6, it is only the average value for the given interval since it assumes rate of change is constant over that interval. Let us move on to the next interval of x = 4 to x = 6. Once again the change in x or Dx equals 2 but the corresponding change in $f(x)$ is not the same as before.

$$\Delta f = f(x+\Delta x) - f(x)$$
$$= (x+\Delta x)^2 - (x)^2$$
$$= x^2 + 2x\Delta x + \Delta x^2 - x^2$$
$$= 2x\Delta x + \Delta x^2$$

For x equal to 4 and Dx = 2, the change in the function over this interval is

$$\Delta f = 2x\Delta x + \Delta x^2$$
$$\Delta f = 2(4)(2) + (2)^2$$
$$\Delta f = 20$$

Note that this corresponds to the same value we would get by :

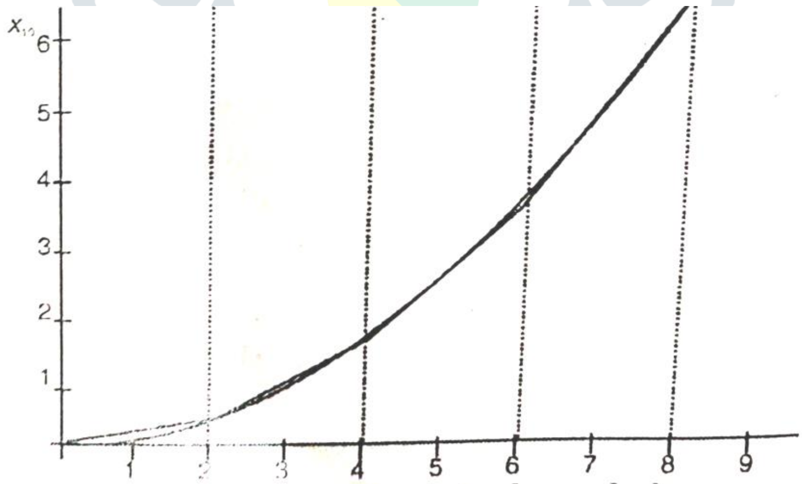
$$\Delta f = f(6) - f(4)$$
$$= 6^2 - 4^2$$
$$= 36 - 16$$
$$= 20$$

The average rate of change over this interval is therefore :

$$\frac{\Delta f}{\Delta x} = \frac{20}{2} = 10$$

This is still larger than the rate of change for the previous interval which was 6. Remember rate of change, by definition, refers to how much the function changes with respect to a change in the independent variable. The steepness or slope of the line over the interval provides a geometric understanding for this concept.

The average rate of changes calculated over each interval can be used to approximate the graph of $f(x)$ from x = 0 to x = 8.



This roughly corresponds to the original graph, but as we can see the rate of change or slope is not constant through out the interval from x = 0 to x = 8, but increases as x-increases. In order to get more accurate answers we need to reduce our interval of Dx = 2 to a much smaller one.

The idea being we need to analyze our graph over a small interval, Dx, to see what exactly is going on at each instant the function is changing. Here is where we begin our study of Calculus. We break down and freeze a changing situation into an infinite series of actions and analyzing what is going on in each individual actions.

IV. Basic Properties of Derivative

We continue our study of classes of functions which are suitably restricted. Again we are passing from the general to the particular. The next most particular class of function we study after the class of continuous functions is the class of differentiable functions. We discuss the definition, show how to get "new functions from old" in what by now is a fairly routine way, and prove that this is a smaller class: that every differentiable function is continuous, but that there are continuous functions that are not differentiable. Informally, the difference is that the graph of a differentiable function may not have any sharp corners in it.

As with continuous functions, our aim is to show that there are many attractive properties which hold for differentiable functions that don't hold in general for continuous functions.

One we discuss in some detail is the ease with which certain limits can be evaluated, by introducing Hospital's rule. Although we don't prove this, the corresponding results are false for continuous functions.

We take the view that much of this material has already been discussed last year, so we move fairly quickly over the basics.

Definition : Let *U* be an open subset of R, and let *f* : U → R. We say that *f* is differentiable at $a \in U$ iff

$$\lim_{x\to\alpha}\frac{f(x)-f(a)}{x-a}$$

or equivalently,

$$\lim_{h\to\alpha}\frac{f(a+h)-f(a)}{h}$$

exists.

The limit, if it exists, is written as $f'(a)$.

We say that *f* is differentiable in *U* if and only if it is differentiable at each *a* in U.

Note that the Newton quotient is not defined when *x* = *a*, not need it be for the definition to make sense. But the Newton quotient, if it exists, can be extended to be a continuous function at *a* by defining its value at the point to be $f'(a)$. Note also the emphasis on the existence of the limit. Differentiation is as much about showing the existence of the derivative, as calculating the value of the derivative.

Example : Let $f(x)=x^3$. Show, directly from the definition, that $f'(a)=3a^2$.

Solution : This is just another way of asking about particular limits, we must compute

$$\lim_{x\to\alpha}\frac{x^3-a^3}{x-a}=\lim_{x\to\alpha}\frac{(x+a)(x^2+xa+a^2)}{x-a}=\lim_{x\to\alpha}(x^2+2xa+a^2)=3a^2$$

Exercise : Let $f(x)=\sqrt{x}$. Show, directly from the definition, that $f'(a)=1/2\sqrt{a}$ when $a\neq 0$. What function do you have to consider in the particular case when $a=4$?

Just as with continuity, it is impractical to use this definition every time to compute derivatives; we need results showing how to differentiate the usual class of functions, and we assume these are known from last year. Thus we assume the rules for differentiation of sums products and compositions of functions, together with the known derivatives of elementary functions such as sin, cos and tan; their reciprocals sec, cosec and cot; and exp and log.

Proposition : Let *f* and *g* be differentiable at *a*, and let *k* be a constant. Then *k*, *f*, *f* + *g* and *fg* are differentiable at *f*. Also, if $ga\neq 0$, then *f/g* is differentiable at *a*. Let *f* be differentiable at *a*, and let *g* be differentiable at *f*(*a*). Then *g* of is differentiable at *a*. In addition, the usual rules for calculating these derivatives apply.

Example : Let $f(x)=\tan\left(\frac{x^2-a^2}{x^2+a^2}\right)$ for $a\neq 0$. Show that *f* is differentiable at every point of its domain, and calculate the derivative at each such point.

Solution : This is the same example we considered in. There we showed the domain was the whole of *R*, and that the function was continuous everywhere. Let $g(x)=\frac{x^2-a^2}{x^2+a^2}$. Then *g* is properly defined for all values of *x*, and the quotient is differentiable, since each term is, and since $x^2 + a^2 \neq 0$ for any x since $a \neq 0$. Thus f is differentiable using the chain rule since $f = \tan og$, and we are assuming known that the elementary functions like tan are differentiable.

Finally to actually calculate the derivative, we have :

$$f'(x) = \sec^2\left(\frac{x^2 - a^2}{x^2 + a^2}\right)\frac{(x^2 + a^2).2x - \left((x^2 - a^2).2x\right)}{(x^2 + a^2)}$$

$$= \frac{4a^2 x}{(x^2 + a^2)^2}.\sec^2\left(\frac{x^2 - a^2}{x^2 + a^2}\right)$$

Exercise : Let $f(x) = \exp\left(\frac{1 + x^2}{1 - x^2}\right)$. Show that f is differentiable at every point of its domain, and calculate the derivative at each such point. The first point in our study of differentiable functions is that it is more restrictive for a function to be differentiable, than simply to be continuous.

Proposition : Let f be differentiable at a. Then f is continuous at a.

Proof : To establish continuity, we must prove that $\lim_{x \to \alpha} f(x) = f(a)$. Since the Newton quotient is known to converge, we have for $x \neq a$,

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}.(x - a) \to f'(a).0 \quad \text{as } x \to a$$

Hence f is continuous at a.

Example : Let $f(x) = |x|$; then f is continuous everywhere, but not differentiable at 0.

Solution : We already know from example that $|x|$ is continuous. We compute the Newton quotient directly; recall that $|x| = x$ if $x \geq 0$, while $|x| = -x$ if $x < 0$. Thus

$$\lim_{x \to 0+} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 0+} \frac{x - 0}{x - 0}, \text{ while}$$

$$\lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{-x - 0}{x - 0} = -1$$

Thus both of the one-sided limits exist, but are unequal, so the limit of the Newton quotient does not exist.

Simple Limits

Our calculus of differentiable functions can be used to compute limits which otherwise prove troublesome.

Proposition : Let f and g be functions such that $f(a) = g(a) = 0$ while $f'(a)$ and $g'(a)$ both exists and $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof : Since $f(a) = g(a) = 0$, provided $x \neq 0$, we have

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x) - f(a)}{x - a}\frac{x - a}{g(x) - g(a)} \to \frac{f'(a)}{g'(a)} as\, x \to a$$

where the last limit exists, since $g'(a) \neq 0$.

Remarks : If $f'(a)$ and $g'(a)$ exist $\lim_{x \to a} \frac{f(x)}{g(x)}$ computing is easy, since f and g must be continuous at a by Proposition, unless we get an indeterminate form 0/0 or ∞ / ∞ for the formal quotient. In fact 1'Hopitals rule helps in both cases, although we need to develop stronger forms.

Example : Show that $\lim_{x \to 0} \frac{3x - \sin x}{x} = 2$.

Solution : Note first that we cannot get the result trivially from, since $g(a)$=0 and so we get the indeterminate form 0/0. However, we are in a position to apply the simple form of 1'Hopital, since $x' = 1 \neq 0$. Applying the rule gives

$$\lim_{x \to 0} \frac{3x - \sin x}{x} = \lim_{x \to 0} \frac{3 - \cos x}{1} = 2.$$

Example : Show that $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = 1/2$.

Solution : Note again that we cannot get the result trivially from 4.16, since this gives the indeterminate 0/0 form, because $g(a)$=0. However, we are in a position to apply the simple form of 1'Hopital, since $x' = 1 \neq 0$. Applying the rule gives

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \to 0} \frac{2^{-1}(1+x)^{-1/2}}{1} = 1/2$$

Example : Show that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Solution : This is spurious because we need the limit to calculate the derivative in the first place, but applying 1'Hopital certainly gives the result.

It has been observed that differential equations can describe any phenomena and the given conditions are completely solvable to find various results.

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