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APPLICATIONS OF DIFFERENTIAL EQUATIONS IN MATHEMATICAL MODELING

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ABSTRACT

The study delves into the analysis of disputes that have emerged between various countries or coalitions, and it presents mathematical ideas and models of international relations as part of the research. A system of differential equations is built upon Richardson's work. You can keep an eye on ODE system stability and solutions simultaneously. Examining the coefficients of the built-model is what makes this job so fascinating. With a focus on mixing problems, this study aims to show how first-order ordinary differential equations may be utilised as a mathematical model to describe some biological processes. In order to simulate a number of biological processes, we evaluated the logistic population model and the prey-predator interaction for three separate species in a linear food chain system using first-order ordinary differential equations. In systems with several tanks, it has also proven useful in addressing some of the issues related to medicine mixing. As was seen in the last section, the logistic model outperforms the exponential model when it comes to population modelling.

Keywords: *mathematical model, logistic model, exponential model*

INTRODUCTION

Various problems in science and engineering may be better understood and solved with the help of differential equations when handled theoretically. In order to comprehend the physical behaviour of the mathematical representation, one must be familiar with its mathematical nature, attributes, and the solution to the differential equation that controls it. In a large part, the rules that govern the universe's operation are claims or relations that pertain to the velocities at which events occur. To formally express this relationship, we utilise equations for the relations and derivatives for the rates. The use of variables, functions, and equations allows us to formally characterise problems before we can find a solution to them in the actual world. These problems are often physical in nature. The issue and its solution may be mathematically represented by an equation like this one. Mathematical modelling is the prescribed procedure for formally creating a model, numerically solving it, and then physically or otherwise understanding the results. A basic building component in most mathematical models is an evolution equation. In the real world, this equation may reveal how certain parts of the situation are likely to evolve.

The description that is used to solve mathematical issues is generated when the model is applied to the description of actual physical behaviours. It is common practice to use equations in mathematical models that connect an unknown function to a derivative or derivatives. This is because rates of change may be quantitatively expressed using derivatives. Equations of this kind are called differential equations. Concurrent ordinary

differential equations are a common solution to problems in many various types of applications when there are several dependent variables, all of which are functions of a single independent variable, most often time. It is not uncommon for the mathematical model that aims to depict a real-life issue to be less complex than the actual event under investigation. The reason for this is because simplifying the assumptions is frequently essential to have a solvable mathematical issue. The use of mathematical ideas demonstrates a basic, hypothetical coin-operated vending machine. This paper presents a mathematical model of thin film flow that solves third-order ordinary differential equations using the numerical solution technique. This scholarly article presents the model. On top of that, several uses for differential equation models are being integrated.

The fundamentals of mathematical modelling, with an emphasis on stability, optimum control, periodic solutions, and qualitative theory in the context of solving differential equations. The exploration of special optimization issues via the use of differential equations as a tool in the domains of engineering and economics. The essay will go over the applications of stochastic and impulsive differential equations, two separate but related types of differential equations. Stieltjes derivatives, used both theoretically and in practice, to first-order ordinary differential equations that have conventional derivatives replaced. Electrode RC constants, chemical reaction kinetics, spectroscopy, relaxation in nuclear magnetic resonance, and the Bouguer-Lambert-Beer law are only a few examples of the many applications of first-order ordinary differential equations. Finding out how first-order differential equations may be used to simulate various types of population dynamics is the main goal of this research. Some examples of these dynamics include development and decay, predator and prey models, and difficulties with mixing in both individual and group tanks.

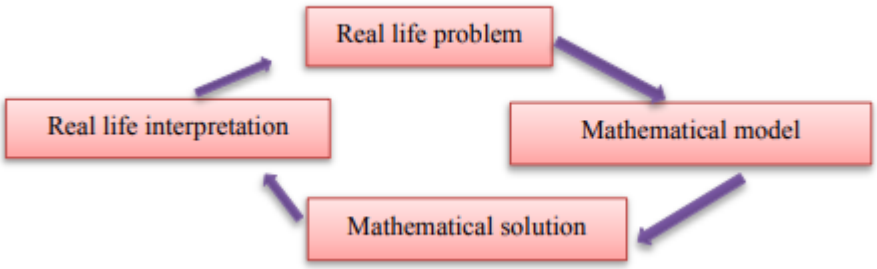
OBJECTIVES

1. Understand the Role of Differential Equations in Modeling Real-World Phenomena

2. Develop Skills in Formulating and Solving Mathematical Models Using Differential Equations

RESEARCH METHODLOGY

Utilising mathematical modelling, it is feasible to expand the scope of the modelling process.



Population law of mass action: It is directly proportional to the rate of change of the first population (b) as a consequence of interaction with the second population (b) that the product of two populations at a certain time b is directly proportional to. Put another way, assuming that there is a proportionality constant $\boldsymbol{V}$, and

$$\frac{dx}{dt} = axy. \quad (1)$$

Balance law for population: The total of the rates of population input into the ecosystem and population outflow during a given time is the net rate of population change, which is indicated as (b). This measure is referred to as the net rate of population change. That is the case

$$\frac{dp}{dt} = \left(\frac{dp}{dt}\right)_{in} - \left(\frac{dp}{dt}\right)_{out}. \quad (2)$$

First order rate law: The growth or decay rate of an initial-order process, denoted by the exponential function (b), is directly proportional to the size of the population at any given instant. With that being said, taking into account the proportionality constant γ,

$$\frac{dp}{dt} = \gamma p(t) \quad (3)$$

As per the principle of conservation of mass, if we are aware of the mass of a substance at a certain point in time, which is represented by the symbol (b), then the equation $dm\ db$ equals zero.

Population Growth or Decay Model

Assuming that the population number of a country at each given time i is denoted by the symbol i, then in accordance with the population balance rule, we may assert that

$$\frac{dp}{dt} = B(p,t) - D(p,t) + M(p,t), \quad (4)$$

a representation of the inputs (birth rates), outputs (death rates), and net migration (m(x,y)) is denoted by the symbol a(x,y). According to model (8), one of the most essential conditions is the assumption that the birth and death rates are perfectly proportionate to the population without foreigners. This is one of the most important situations. Therefore,

$$B(p,t) = bp(t), \quad D(p,t) = dp(t), \quad M(p,t) = 0 \quad (5)$$

This enables us to reduce the complexity of the equation to

$$\frac{dp}{dt} = (b-d)p = \gamma p, \quad (6)$$

This indicates that the population is increasing when the value of γ is more than zero, and it is falling when the value of γ is less than zero. The representation of the proportionality constant looks like this: $b - d = \gamma$. We are able to solve problem (9) in the following manner because to the fact that it is a linear differential equation:

$$p(t) = p_0 e^{\gamma t}. \quad (7)$$

The initial population is represented by the equation ($b0$) = $k0$, while the growth or decay constant is indicated by the symbol γ. As a result, when the value of γ is greater than zero, the population continues to increase until it reaches infinity. Conversely, when γ is less than zero, the population declines and approaches zero. The difficulty is that there are limitations to how large a population can develop before there is competition for things like space, resources, and food. This is the most significant limitation. If a certain ecosystem is only capable of supporting a particular number of people, we will refer to it as a Z. The environmental carrying capacity is the numerical value that is referred to as the quantity x. The conclusion that can be drawn from this is that the equation (9) is likely to decrease for other models whenever the size of the population B increases. It may be

stated in a number of different ways, but one way to describe it is that the rate of population increase (or fall) is unaffected by time-dependent elements such as seasonal occurrences and is solely controlled by the number of populations.

$$\frac{dp}{dt} = pf(p). \tag{8}$$

Then, let's pretend that (p) is a linear function.

$$f(p) = \alpha p + \beta \tag{9}$$

with conditions

$$\lim_{p(t)\to 0} fp(t) = \gamma, f(k) = 0 \text{ which leads } f(p) = \gamma - \left(\frac{\gamma}{k}\right)p. \tag{10}$$

Equation (10) becomes

$$\frac{dp}{dt} = p\left(\gamma - \frac{\gamma}{k}p\right) \tag{11}$$

This is the name given to the logistic population model that takes into account both the growth rate γ and the carrying capacity k. It is evident that the equation may be simplified to the nonlinear, separable exponential form when we assume that the value of (b) is relatively low in comparison to The. A collection of constant solutions is the equilibrium solution. These solutions are those in which r = 0 and r = −1. From equation (11), we may be able to,

$$p(t) = \frac{kp_0}{p_0 + (k - p_0)e^{-\gamma t}} \tag{12}$$

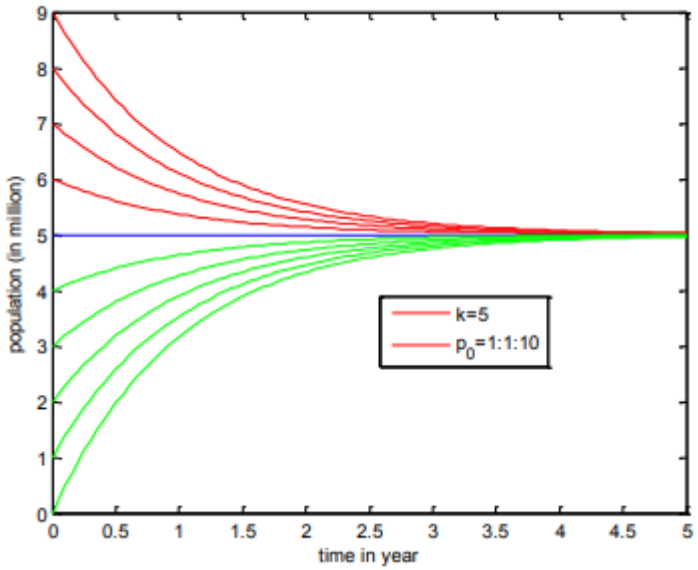


Fig 1: logistic population model with γ = 1

Prey predator model

This model provides a qualitatively accurate description of a linear feeding chain consisting of three different species. Imagine a scenario in which The same habitat supports three distinct animal species. In an ideal world, there would be a simple three-tiered food chain in which one species of predator would eat another species of prey at the middle level (M), and so on. Examples of ecosystems that include three species were provided by Paullet et al. (2002). These ecosystems included the mouse-snake-owl and the worm-robin-falcon. The sole components that make up the predator-prey association paradigm are natural processes such as development and decay, as well as the interaction between the predator and the prey. The other connections (factors) are not taken into consideration in this research as if they did not exist. The population of prey is said to rise when there are no predators present, in accordance with a first-order rate law, and the same rule states that the presence of predators causes the population of prey to fall. If there were a one-to-one relationship between the number of prey species and their rate of growth, then the first order rate rule would hold. Assuming there are no predators and an infinite supply of food, this is the underlying premise.

$$\frac{dx}{dt} = ax, \qquad a > 0 \tag{13}$$

Conversely, the presence of predator species causes a drop in the population of prey species equal to bxy, where b is a positive integer. This suggests that the pace at which predators consume prey has a negative effect on the prey population. After plugging this rate into equation (13), the following model for the prey species population may be constructed:

$$\frac{dx}{dt} = ax - bxy \tag{14}$$

It is plausible to suppose that the population of middle-level species would decline if the environment did not include any prey species. For the simple reason that:

$$\frac{dy}{dt} = -cy \qquad c > 0 \tag{15}$$

If predator and prey populations are both present in an ecosystem, then the product of their populations is the frequency with which the two species interact or meet each other during a particular time period. Given the presence of prey species, the recruitment rate of mid-level species to the system is r, which is non-zero, since there is a food source. Regardless, mid-level species populations are reduced by a factor of bbz when top-level predators are present (where b is non-zero). The rate of predation that mid-level species face when interacting with top-level predators further worsens this reduction: You may get a model for the mid-level species population by plugging this rate into equation (15):

$$\frac{dy}{dt} = -cy + exy - gyz. \tag{16}$$

It is also possible to find a model for the population of top-level species as

$$\frac{dz}{dt} = -hz + lyz\ , l > 0. \tag{17}$$

Representing a system of ordinary differential equations that are not linear is the set of equations (14) through (17). We next suggested studying the model, which turns out to be

$$\frac{dx}{dt} = ax - bxy$$
$$\frac{dy}{dt} = -cy + exy - gyz$$
$$\frac{dz}{dt} = -hz + lyz \quad (18)$$

Because populations do not have negative values, x, y, and z may also have nonnegative values. Without a mid-level predator y, the prey's natural growth rate is a. Predation's impact on prey x is represented by b. Without prey x, the mid-level predator y's natural mortality rate is represented by f. When prey x is not available, the mid-level predator y's propagation rate and efficiency are denoted by b. The impact of species z's predation on species y is represented as (1, -1). In a predator-free environment, the natural mortality rate of z is represented as h. n - the rate of proliferation and effectiveness of predator z when prey is present. When there is no top predator ($\boldsymbol{E}$ = 0), the model simplifies to:

$$\begin{cases} \frac{dx}{dt} = ax - bxy \\ \frac{dy}{dt} = -cy + exy \end{cases}$$

whose shape the answer takes on in the xy plane:

$$y^a x^c = Ee^{ex+by}$$

where the integration constant is denoted by . When species at the intermediate level are not present (y = 0), At its core, the model is:

$$\begin{cases} \frac{dx}{dt} = ax \\ \frac{dz}{dt} = -hz \end{cases}$$

that provides $z = Mx^{\frac{-h}{a}}$ in terms of a constant of integration M

The model is reduced to the following when there are no bottom-level prey species (x = 0):

$$\begin{cases} \frac{dy}{dt} = -cy - gyz \\ \frac{dz}{dt} = -hz + lyz \end{cases}$$

In the yz plane, there is a solution for some integration constant N, in the form of $z\ cy\ -h\ =\ Ne\ -bz-ly$. Equational equilibrium points are the algebraic system's solutions. (18).

$$\begin{cases} (a-by)x=0 \\ (-c+ex-gz)y=0. \\ (-h+ly)z=0 \end{cases}$$

RESULT

So, there are three points of equilibrium ($x0$, $y0$, $z0$) at (0,0,0), (c/e, a/b, 0) and (0, h/l,$-c/g$). One of the equilibrium points ($x0$, $y0$, $z0$) may be used to linearise system (18), as In x, y, and z, the right-hand sides of the system possess continuous partial derivatives. The resulting linearised system is represented by

$$\begin{cases} \dfrac{dx}{dt} \approx (a-by_0)(x-x_0)-bx_0(y-y_0) \\ \dfrac{dy}{dt} \approx ey_0(x-x_0)+(-c+ex_0-gz_0)(y-y_0)-gy_0(z-z_0) \\ \dfrac{dz}{dt} \approx lz_0(y-y_0)+(-h+ly_0)(z-z_0) \end{cases}$$

It is represented by the Jacobian matrix

$$J(x_0,y_0,z_0)=\begin{bmatrix} a-by_0 & -bx_0 & 0 \\ ey_0 & -c+ex_0-gz_0 & -gy_0 \\ 0 & lz_0 & -h+ly_0 \end{bmatrix}. \quad (19)$$

At the point of equilibrium (0, 0, 0), the form of the Jacobian matrix (19) is

$$J(0,0,0)=\begin{bmatrix} a & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & -h \end{bmatrix}$$

The vectors that correspond to the eigenvectors are ⟨1,0,0□, ⟨0,1,0□, and ⟨0,0,1□. On the other hand, the eigen values are $\lambda 1 = n$, $\lambda 2 = -c$, and $\lambda 3 = -h$. Due to the fact that X is an eigenvalue that has a positive real part, the point of equilibrium (0,0,0) is vulnerable to instability. The general solution is stated using the eigenvalue and eigenvector in a linear fashion such that correspond to it. This is the general solution being expressed. When put another way,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}e^{at}+c_2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}e^{-ct}+c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}e^{-ht}$$

where $c1$, $c2$, $c3$ constants that may be determined arbitrarily. What this generic solution does in the long run is

$$\text{As } \ t\to\infty, \ \begin{cases} x(t)\to\infty \\ y(t)\to 0 \\ z(t)\to 0 \end{cases}$$

If the mid-level and top-level predators were to be eliminated from this simplified model, the prey population would increase according to the physical interpretation of this reaction. Based on this response, we can deduce this. The Jacobian matrix (19) has the form that is attained when it is at the point of equilibrium c/e, a/b.

$$J(c/e, a/b, 0) = \begin{bmatrix} 0 & \frac{-cb}{e} & 0 \\ \frac{ae}{b} & 0 & \frac{-ag}{b} \\ 0 & 0 & \frac{-hb+al}{b} \end{bmatrix}$$

with eigen values

$$\lambda_1 = \frac{la-hb}{b}, \qquad \lambda_2 = -i\sqrt{ac}, \qquad \lambda_3 = i\sqrt{ac}$$

and corresponding eigenvectors respectively:

$$\langle 1, \frac{(hb-al)e}{b^2c}, \frac{ab^2ce + b^2eh^2 - 2abehl + a^2el^2}{ab^2cg} \rangle \ , \langle 1, \frac{ie\sqrt{ac}}{bc}, 0 \rangle, \ \langle 1, \frac{-ie\sqrt{ac}}{bc}, 0 \rangle$$

Because of this, it can be deduced that the equilibrium point is stable when the difference between la and hb is less than zero, and it is unstable when the difference between la and hb is more than zero. There are three eigen values that have zero real part that make up when la equals 0, the Jacobian matrix is assessed at this equilibrium point. Taking this into consideration, each and every equilibrium point is a solid surface. This point of equilibrium must be maintained in a steady state until further notice. As a result of the stability of the equilibrium point, which draws populations that are not zero, the dynamics of the system have the potential to bring about the extinction of all three species, particularly for different beginning population levels. To summarise, the solution to this question is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \frac{(hb-al)e}{b^2c} \\ \frac{ab^2ce + b^2eh^2 - 2abehl + a^2el}{ab^2cg} \end{pmatrix} e^{\left(\frac{la-hb}{b}\right)t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cos\sqrt{act}\, t - \begin{pmatrix} 0 \\ \frac{e\sqrt{ac}}{bc} \\ 0 \end{pmatrix} \sin\sqrt{act}\, t \right]$$
$$+ c_3 \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \sin\sqrt{act}\, t + \begin{pmatrix} 0 \\ \frac{e\sqrt{ac}}{bc} \\ 0 \end{pmatrix} \cos\sqrt{act}\, t \right]$$

in where c_1,c_2,c_3 are positive constants that may be anything. The following table shows the general solution's behaviour over the long run. Because $la - ab$ is greater than zero

As $t \to \infty$	$x(t) \to \infty$
	$y(t) \to -\infty$
	{ $z(t) \to \infty$, if $ab\ 2\ ce + b\ 2\ eh\ 2 - 2abehl + a\ 2\ el > 0$ $z(t) - \infty$, if $ab\ 2\ ce + b\ 2\ eh\ 2 - 2abehl + a\ 2\ el < 0$

CONCLUSION

A description of how to simulate real-world events using first-order ordinary differential equations was the intended goal of this work. In both single- and multi-tank systems, the software incorporates models of population growth and decay, predator-prey interactions, and mixing problems. In addition, there are mixing issues with the bundle. We conclude that, to some extent, systems of non-linear order ordinary differential equations may provide a satisfactory explanation for the mathematical model of prey-predator interactions. If the exponential model has an issue, the logistic model can fix it. According to the exponential model, a population may either continue to grow indefinitely or eventually die out. There is room for both of these possibilities. Conflict over finite resources and physical limitations are two factors that restrict how quickly a population may expand. A logistic model may account for this factor, for example, by assuming a population-to-growth ratio. Furthermore, it has been shown that a well-mixed solution may be obtained by collecting its concentration after a certain period of time. Finally, the article argued that ordinary differential equations are the key to solving many future technological problems.

REFERENCES

[1] Agarap, A.F. (2016). A Mathematical Model of a Hypothetical Coin-Operated Vending Machine Designed Using Concepts in Automata Theory.

[2] Ahsan, Z. (2016). Differential equations and their applications. PHI Learning Pvt. Ltd.

[3] Bajpai, A.C., Mustoe, L.R. and Walker, D. (2018). Advanced engineering mathematics.

[4] Bellomo. N, Angelis. E and Delitala. M. (2007). Lecture Notes on Mathematical Modeling in Applied Sciences.

[5] Boyce, W.E., DiPrima, R.C. and Meade, D.B. (2017). Elementary differential equations. John Wiley & Sons.

[6] Cesari, L. (2012). Optimization—theory and applications: problems with ordinary differential equations. Springer Science & Business Media. Vol. 17.

[7] Edwards, C.H., Penney, D.E. and Calvis, D.T. (2016). Differential equations and boundary value problems. Pearson Education Limited.

[8] Elayedi. S. (2005). an introduction to difference equations, springer science and business media Inc., third edition.

[9] Frigon, M. and Pouso, R.L. (2017). Theory and applications of first-order systems of Stieltjes differential equations. Advances in Nonlinear Analysis, 6(1): 13-36.

[10] Jeffrey. A. (2010). matrix operations for engineers and scientists, e studio calamar s. l. Germany, first edition.

[11] Jordan .D and Smith. P. (2007). Nonlinear Ordinary Differential Equations, Oxford University Press Inc., fourth edition.

[12] Kolmanovskii, V. and Myshkis, A. (2013). Introduction to the theory and applications of functional differential equations. Springer Science & Business Media. Vol. 463.

[13] Lambe, C.G. and Tranter, C.J. (2018). Differential equations for engineers and scientists. Courier Dover Publications.

[14] Logan, D. (2017). A first course in differential equations. Springer.

[15] Mechee, M., Senu, N., Ismail, F.,

[16] Nikouravan, B. and Siri, Z. (2013). A three-stage fifth-order Runge-Kutta method for directly solving special third-order differential equation with application to thin film flow problem. Mathematical Problems in Engineering.

[17] Oksendal, B. (2013). Stochastic differential equations: an introduction with applications. Springer Science & Business Media.

[18] Paullet .J, Previte. J and Walls. Z. (2002). lotka -volterra three species food chain, 75: 243.

[19] Scholz, G. and Scholz, F. (2015). First-order differential equations in chemistry. ChemTexts, 1(1): 1.

[20] Simeonov, P. (2007). Impulsive differential equations: periodic solutions and applications.

[21] Routledge. Simmons, G.F. (2016). Differential equations with applications and historical notes. CRC Press.

[22] Slavik. A. (2013). Mixing Problems with Many Tanks.

[23] Zill. D. (2013). A first course in differential equations with modeling applications, Brooks/cole, cengage learning, tenth edition