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FEKETE SZEGO COEFFICIENT INEQUALITY OF REGULAR FUNCTIONS FOR A SPECIAL CLASS

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ABSTRACT: We will consider a new type of family of analytic functions and its subclasses will be discussed here, by which coefficient bounds of Fekete Szego functional $|a_3 - \mu a_2^2|$ for the analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ fitting in these classes and subclasses, will be obtained.

KEYWORDS: Univalent functions, Coefficient inequality, Starlike functions, Convex functions, Close to convex functions and bounded functions.

MATHEMATICS SUBJECT CLASSIFICATION: 30C50

1. Introduction : Let $\mathcal{A}$ denote the family of functions of the type

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

regular in the unit disc $\mathbb{E} = \{z :|z|< 1|\}$. Let the family of functions of the form (1.1) which are analytic and univalent in $\mathbb{E}$ be denoted by $\mathcal{S}$,.

Bieber Bach ([7], [8]) proved in 1916,that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. Löwner [5] proved in 1923,that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the recognized estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, naturally some relation was to be sought between a_3 and ${a_2}^2$ for the class $\mathcal{S}$,Löwner's method was used by Fekete and Szegö [9] to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if\ \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \leq \mu \leq 1; \\ 4\mu - 3, if \mu \geq 1. \end{cases} \tag{1.2}$$

The inequality (1.2) plays a crucial role in determining approximations of higher order coefficients for some subclasses $\boldsymbol{\mathcal{S}}$ (See Chhichra [1], Babalola [6]).

Let us outline some subclasses of $\boldsymbol{\mathcal{S}}$.

We will denote by S*, the family of univalent and starlike functions

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \boldsymbol{\mathcal{A}}$ and satisfying the condition

$$Re\left(\frac{zg'(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \tag{1.3}$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \boldsymbol{\mathcal{A}}$$

and satisfying the condition

$$Re\frac{((zh'(z))'}{h'(z)} > 0, z \in \mathbb{E}. \tag{1.4}$$

A function $f(z) \in \boldsymbol{\mathcal{A}}$ is known as close to convex function if there exists $g(z) \in S^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \tag{1.5}$$

Kaplan [3] familiarized us with the class of close to convex functions and denoted it by C and proved that all close to convex functions are univalent.

We introduced a new subclass

$$\left\{f(z) \in \mathcal{A}; \frac{z[(f'(z))^2 + f(z)f"(z)]}{f(z)f'(z)} \prec \left(\frac{1+Az}{1+Bz}\right)^{\delta}; z \in \mathbb{E}\right\}$$

and we will denote it as $\boldsymbol{S^*(f, f', f'', A, B, \delta)}$.

Symbol $\prec$ stands for subordination, which we describe as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$; $z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form

$$w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1. \tag{1.8}$$

It is known that

$$|d_1| \le 1, |d_2| \le 1 - |d_1|^2. \tag{1.9}$$

2. **PRELIMINARY LEMMAS:** For $0 < c < 1$, we write

$$w(z) = \left(\frac{c+z}{1+cz}\right)$$

so that

$$\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\delta} = 1 + (A-B)\delta c_1 z + (A-B)\delta(c_2 - B\delta c_1^2)z^2 + --- \quad (2.1)$$

3. MAIN RESULTS

THEOREM 3.1: Let $f(z) \in \boldsymbol{S}^*(\boldsymbol{f}, \boldsymbol{f}', \boldsymbol{f}'', \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{\delta})$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(A-B)\delta[\delta(5A-14B)]}{72} - \frac{\delta^2(A-B)^2}{9}\mu \; ; if \; \mu \leq \frac{\delta(5A-14B)-9}{8\delta(A-B)} & (3.1) \\ \frac{\delta(A-B)}{8} ; if \; \frac{\delta(5A-14B)-9}{8\delta(A-B)} \leq \mu \leq \frac{\delta(5A-14B)+9}{8\delta(A-B)} & (3.2) \\ \frac{\delta^2(A-B)^2}{9}\mu - \frac{\delta(A-B)[\delta(5A-14B)]}{72} \; ; if \; \mu \geq \frac{\delta(5A-14B)+9}{8\delta(A-B)} & (3.3) \end{cases}$$

The results are sharp.

Proof: By definition of $f(z) \in S_n^*(A;B)$, we have

$$\frac{z[(f'(z))^2 + f(z)f''(z)]}{f(z)f'(z)} = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^{\delta} ; w(z) \in \mathcal{U}. \quad (3.4)$$

Expanding the series (3.4), we get

$$\{1 + 6a_2 + (6a^2{}_2 + 12a_3)z^2 + \cdots\}$$
$$= \{1 + [(A-B)\delta c_1 + 3a_2]z + [\delta(A-B)(c_2 - B\delta c_1^2) + 3a_2(A-B)\delta c_1 + 4a_3 + 2a^2{}_2]z^2 + \cdots\} \quad (3.5)$$

Identifying terms in (3.5), we get

$$a_2 = \frac{(A-B)\delta}{3} c_1 \quad (3.6)$$

$$a_3 = \frac{\delta(A-B)}{8} c_2 + \frac{\delta^2(A-B)(5A-14B)}{72} c_1^2 \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_3 - \mu a_2^2 = \frac{\delta(A-B)}{8} c_2 + \frac{\delta^2(A-B)}{2}\left\{\frac{(5A-14B)}{72} - \frac{(A-B)}{9}\mu\right\} c_1^2 \quad (3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{8}|c_2| + \frac{\delta^2(A-B)}{2}\left|\frac{(5A-14B)}{72} - \frac{(A-B)}{9}\mu\right||c_1|^2. \quad (3.9)$$

Using (1.9) in (3.9), we get

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{8}(1-|c_1|^2) + \frac{(A-B)}{2}\left|\frac{(A-2B)}{3^n} - \frac{(A-B)}{2^{2n-1}}\mu\right||c_1|^2$$

$$= \frac{\delta(A-B)}{8} + \left\{\left|\frac{\delta^2(A-B)(5A-14B)}{72} - \frac{\delta^2(A-B)^2}{9}\mu\right| - \frac{\delta(A-B)}{8}\right\}|c_1|^2 \quad (3.10)$$

Case I: $\mu \leq \frac{9(5A-14B)}{8(A-B)}$

(3.10) can be rewritten as

$$\left|a_3 - \mu a_2^2\right| \leq \frac{\delta(A-B)}{8} + \left\{\frac{\delta(A-B)[\delta(5A-14B)-9]}{72} - \frac{\delta^2(A-B)^2}{9}\mu\right\}|c_1|^2 \quad (3.11)$$

Subcase I (a): $\mu \leq \frac{[\delta(5A-14B)-9]}{\delta.8(A-B)}$

Using (1.9), (3.11) becomes

$$\left|a_3 - \mu a_2^2\right| \leq \frac{\delta(A-B)[\delta(5A-14B)]}{72} - \frac{\delta^2(A-B)^2}{9}\mu \quad (3.12)$$

Subcase I (b): $\mu \geq \frac{[\delta(5A-14B)-9]}{\delta.8(A-B)}$

We obtain from (3.11)

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{8} \quad (3.13)$$

Case II: $\mu \geq \frac{9(5A-14B)}{8(A-B)}$

Preceding as in case I, we get

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{8} + \left\{\frac{\delta^2(A-B)^2}{9}\mu - \frac{\delta(A-B)[\delta(5A-14B)+9]}{72}\right\}|c_1|^2 \quad (3.14)$$

Subcase II (a): $\mu \leq \frac{\delta(5A-14B)+9}{8\delta(A-B)}$

(3.14) takes the form

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{8} \tag{3.15}$$

Combining the results of subcases I(b) and II(a), we can write

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{8} \; ; \; if \; \frac{[\delta(5A-14B)-9]}{\delta.8(A-B)} \leq \mu \leq \frac{\delta(5A-14B)+9}{8\delta(A-B)} \tag{3.16}$$

<u>Subcase II (b)</u>: $\mu \geq \frac{\delta(5A-14B)+9}{8\delta(A-B)}$

Preceding as in subcase I (a), we get

$$\left|a_3 - \mu a_2^2\right| \leq \frac{\delta^2(A-B)^2}{9}\mu - \frac{\delta^2(A-B)(5A-14B)}{72} \tag{3.17}$$

<u>Combining (3.12), (3.16) and (3.17), the theorem is established</u>.

Extremal function for (3.1) and (3.3) is demarcated by

$$f_1(z) = z\left\{1 + \frac{p^2}{(p^2 - 2q)}\right\}^{\frac{p^2-2q}{p}}$$

Extremal function for (3.2) is defined by

$$f_2(z) = z(1 + z^2)^q$$

Where $p = \frac{\delta(A-B)}{3}$ and $q = \frac{(A-B)\delta[\delta(5A-14B)]}{72}$

Corollary 3.2: Putting $A = 1, B = -1 \; and \; \delta = 1$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{19}{36} - \frac{4}{9}\mu, if \mu \leq \frac{5}{8}; \\ \frac{1}{4} \quad if \frac{5}{8} \leq \mu \leq \frac{7}{4}; \\ \frac{4}{9}\mu - \frac{19}{36}, if \; \mu \geq \frac{7}{4} \end{cases}$$

These approximations were derived by G. Singh [6] and are outcomes for the class of univalent functions.

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