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Hankel and Toeplitz Determinant Inequalities of Third Order for Certain Analytic Univalent Functions Related to Four-Leaf Function

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Abstract
The aim of this paper is to compute upper bound for third Hankel and Toeplitz determinants for certain analytic univalent functions associated to four-leaf function defined on the open unit disk. Suitable examples have been provided in support of our results.

Keywords: *Analytic univalent function; Four-leaf function; Hankel determinant; Toeplitz determinant;*

1. INTRODUCTION

Let $\mathcal{A}$ be the family of analytic functions f defined on the open unit disk Δ in the complex plane $\mathbb{C}$ satisfying the normalization $f(0) = 0 = f'(0) - 1$. The collection of univalent functions $f \in \mathcal{A}$ is denoted by S. The well-known classes of starlike, convex and bounded turning functions are respectively denoted by S^*, C and R are some subclasses of S. The Maclaurin series expansion of f be of the form

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for all $z \in \Delta$. (1.1).

In 1959, Sakaguchi [5] introduced the class S_s^* of starlike functions with respect to symmetrical points. A function $f \in A$ is said to be starlike with respect to symmetric points if and only if,

$$Re\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) > 0, \text{for } z \in \Delta$$

In 1977, Das and Singh [6] defined the class C_s of convex functions with respect to symmetric points. A function f A is said to be convex with respect to symmetric points if and only if

$$Re\left(\frac{2(zf'(z))'}{(f(z) - f(-z))'}\right) > 0, \text{for } z \in \Delta$$

In 1976, Noonan and Thomas [51] defined qth Hankel determinant of index $n \geq 1$ for $f \in A$ denoted by $H_{q,n}(f)$ or $H_q(n)$,which is stated by (see [7])

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (1.2)$$

Whereas qth symmetric Toeplitz determinant$T_q(n)$, for f ∈A defined as(see[8]),

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix} \qquad (1.3)$$

The upper bounds of Hankel and Toeplitz determinants associated with coefficients of analytic functions with certain interesting geometric properties have been discussed by several authors in recent past.

2. Literature Review and Motivation.

The study of Hankel and Toeplitz determinants associated with $f \in S$play a vital role in the field of Geometric Function theory. The pioneering works of Pommerenke, Hayman, Babalola, Zaprawa, Kowalezyk and Oh Sang Kown concerning Hankel determinants for $f \in S$ motivated numerous other researchers to investigate $|H_2(2)|$ and $|H_3(1)|$ for various other subclasses of S (see [9] and references therein).

Recently, Gandhi ([10]) introduced and studied radius estimates for $f \in S$ subordinate to $\varphi_{3L}(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4$which maps Δ onto the interior of three leaf domain. Further, Lei Shi et al. and Murugusundaramoorthy et al. studied upper bound of $|H_3(1)|$ for f in $S^*(\varphi_{3L})$and $R(\varphi_{3L})$ respectively. (see ([11]) and [12]). The classes

$$S^*_{4L} = \left\{ f \in S : \frac{zf'(z)}{f(z)} \prec \varphi_{4L}(z) \right\} ,$$

$$R_{4L} = \{ f \in S : f'(z) \prec \varphi_{4L}(z) \}$$

of starlike and bounded turning functions are associated with $\varphi_{4L}(z) = 1 + \frac{5}{6}z + \frac{1}{6}z^5$ that maps Δ onto four leaf shaped domain have been introduced by Pongsakorn Sunthrayuth et al. [13]. Kumar et al studied Hankel and symmetric Toeplitz determinants for Sakaguchi starlike functions [14]. Bharavi Sharma et al studied third Hankel Determinant for a class of functions with respect to symmetric points associated with exponential function [16]

Motivated by the earlier mentioned research work, in this paper, upper bounds of third order Hankel determinant $|H_{3,1}(f)|$and third order Toeplitz determinant$|T_3(1)|$ for$f \in C_s(\varphi_{4L})$ were computed, where

$$C_s(\varphi_{4L}) = \left\{ f \in A : \frac{2\left(zf'(z)\right)'}{\left(f(z) - f(-z)\right)'} \prec \varphi_{4L}(z) \right\}$$

The image of Δ under the mapping $\varphi_{4L}(z)$ is shown below. (Generated using Complex Tools software)

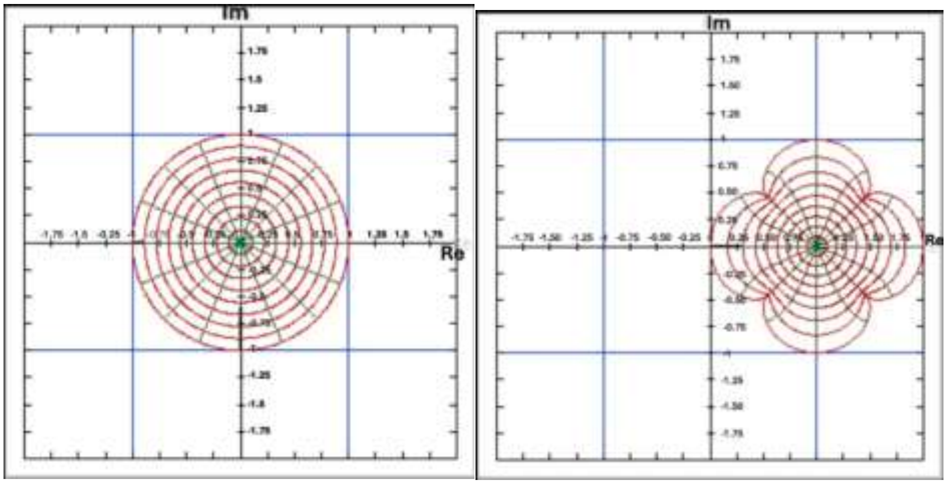

Figure 1. The Image of Δ under $\varphi_{4L}(z)$

2.1. A Set of Useful Lemmas

The collection $\mathcal{P}$ consists of analytic functions $p: \Delta \to \mathbb{C}$ with $p(0) = 1$ *and* $\Re\{p(z)\} > 0$. The following lemmas concerning $p \in \mathcal{P}$ of the form

$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ *for all* $z \in \Delta$ ……. (2.1)

are useful in proving the main result.

Lemma 2.1. ([1]) If $p \in \mathcal{P}$ is of the form (2.1), then $|c_n| \le 2$ for any positive integer n. The inequality is sharp for $p(z) = \frac{1+z}{1-z}$.

Lemma 2.2.([2]) If $p \in \mathcal{P}$ is of the form (2.1), then for any μ ∈ C, $\left|c_2 - \mu c_1^2\right| \le 2\max\{1, |2\mu - 1|\}$, and the inequality is sharp for $p(z) = \frac{1+z}{1-z}$ and $p(z) = \frac{1+z^2}{1-z^2}$.

Lemma 2.3. ([3]) If $p \in \mathcal{P}$ is of the form (2.1), then

$\left|Jc_3 - Kc_1c_2 + Lc_3\right| \le 2|J| + 2|K - 2J| + 2|J - K + L|$ for real numbers J, K and L.

Lemma 2.4. ([4]) If $p \in \mathcal{P}$ is of the form (2.1), then for all $n, m \in N$, the set of positive integers, $\left|\mu c_n c_m - c_{n+m}\right| \le 2|2\mu - 1|$. This inequality is sharp.

Lemma 2.5. ([4]) If m, n, l, r and the inequalities $0 < m < 1$, $0 < r < 1$ and

$8r(1-r)\left((mn-2l)^2 + (m(r+m)-n)^2\right) + m(1-m)(n-2rm)^2 \le 4m^2(1-m)^2 r(1-r)$ hold. If

If $p \in \mathcal{P}$ is of the form (2.1), then $\left|lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3n}{2}c_1^2c_2 - c_4\right| \le 2$.

3. Main Results

Let $f \in C_s(\varphi_{4L})$. Then there exists $w \in \mathcal{B}$ such that

$$\frac{2(zf'(z))'}{(f(z)-f(-z))'} = \varphi_{4L}(w(z)), for\ all\ z \in \Delta \text{........(3.1).}$$

If we take $p(z) = \frac{1+w(z)}{1-w(z)} \in \mathcal{P}\ for\ all\ z \in \Delta$, then $w(z) = \frac{p(z)-1}{p(z)+1}$

so that $\frac{2(zf'(z))'}{(f(z)-f(-z))'} = 1 + \frac{5}{6}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{1}{6}\left(\frac{p(z)-1}{p(z)+1}\right)^5, for\ all\ z \in \Delta$ ....(3.2)

But, $\frac{2(zf'(z))'}{(f(z)-f(-z))'} = 4a_2z + 6a_3z^2 + (16a_4 - 12a_2a_3)z^3 + (20a_5 - 18a_3^2)z^2$.......(3.3)

$$1 + \frac{5}{6}\left(\frac{p(z)-1}{p(z)+1}\right) + \frac{1}{6}\left(\frac{p(z)-1}{p(z)+1}\right)^5 = 1 + \frac{5}{6}\left[\frac{1}{2}c_1z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{1}{8}c_1^3 - \frac{1}{2}c_1c_2 + \frac{1}{2}c_3\right)z^3 + \left(\frac{1}{2}c_4 - \frac{1}{2}c_1c_3 - \frac{1}{4}c_2^2 - \frac{1}{16}c_1^4 + \frac{3}{8}c_1^2c_2\right)z^4\right] \text{...... (3.4)}$$

On substituting (3.3) and (3.4) in (3.2) and comparing like coefficients on both the sides of (3.2), we obtain

$a_2 = \frac{5}{48}c_1$...... (3.5)

$a_3 = \frac{5}{72}\left(c_2 - \frac{1}{2}c_1^2\right)$......... (3.6)

$a_4 = \frac{5}{192}\left(\frac{7}{48}c_1^3 - \frac{19}{24}c_1c_2 + c_3\right)$......... (3.7)

$a_5 = \frac{-5}{240}\left(\frac{7}{96}c_1^4 + \frac{7}{24}c_2^2 + c_1c_3 - \frac{13}{24}c_1^2c_2 - c_4\right)$......... (3.8).

Example 3.1. By taking the Schwarz functions $w(z) = z, w(z) = z^2, w(z) = z^3, w(z) = z^4$ in (3.1) followed by integrating on both sides and utilizing the fact $f(0) = f'(0) = 1$, we get respectively.

(1) $f_1(z) = z + \frac{5}{24}z^2 + \frac{1}{216}z^6 + \cdots$

(2) $f_2(z) = z + \frac{5}{36}z^3 + \frac{25}{1,440}z^5 + \cdots$

(3) $f_3(z) = z + \frac{5}{96}z^4 + 0z^9 + \cdots$

(4) $f_4(z) = z + \frac{5}{120}z^5 + \frac{25}{10,368}z^9 + \cdots$

It is easy to see that $f \in C_s(\varphi_{4L})$for i = 1, 2, 3, 4.

We now estimate initial coefficient bounds for the functions in $C_s(\varphi_{4L})$.

Theorem 3.1. If $f \in C_s(\varphi_{4L})$ be of the form (1.1), Then

$|a_2| \leq \frac{5}{24}, |a_3| \leq \frac{5}{36}, |a_4| \leq \frac{5}{96}$ and$|a_5| \leq \frac{1}{24}$.

The functions f_1, f_2, f_3, f_4 as given in the Example 3.1 are extremal functions for these inequalities respectively.

Proof. Let $f \in C_s(\varphi_{4L})$ be given by (1.1). By applying Lemmas 2.1, 2.2 and 2.3 to (3.5), (3.6) and (3.7) respectively, we obtain

$$|a_2| = \frac{5}{48}|c_1| \leq \frac{5}{24}$$

$$|a_3| = \frac{5}{72}\left|c_2 - \frac{1}{2}c_1^2\right| \le \frac{5}{36}$$

$$|a_4| = \frac{5}{192}\left|\frac{7}{48}c_1^3 - \frac{19}{24}c_1c_2 + c_3\right| \le \frac{5}{96}$$

By taking modulus on both sides of (3.8), we get

$$|a_5| = \frac{5}{240}\left|\frac{7}{96}c_1^4 + \frac{7}{24}c_2^2 + c_1c_3 - \frac{13}{24}c_1^2c_2 - c_4\right|$$

$$|a_5| = \frac{1}{48}\left|lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3n}{2}c_1^2c_2 - c_4\right|$$

Where $l = \frac{7}{96}, r = \frac{7}{24}, m = \frac{1}{2}, n = \frac{13}{26}$.

The values of l, r, m, n satisfy the inequality in the hypotheses of Lemma (2.5) and hence $|a_5| \le \frac{1}{48}(2) = \frac{1}{24}$.

Theorem 3.2.If $f \in C_s(\varphi_{4L})$ be of the form (1.1), Then for any $\rho\epsilon\mathbb{C}$, we have $|a_3 - \rho a_2^2| \le \frac{5}{36} max\left\{1, \frac{5}{16}|\rho|\right\}$ and this inequality is sharp.

Proof: Let $f \in C_s(\varphi_{4L})$. Then in view of Lemma 2.2, we obtain

$$|a_3 - \rho a_2^2| = \left|\frac{5}{72}\left(c_2 - \frac{1}{2}c_1^2\right) - \rho\left(\frac{25}{2{,}304}\right)c_1^2\right|$$

$$\le \frac{5}{72}(2)max\left\{1, \left|\frac{16+5\rho}{16} - 1\right|\right\}$$

$$\Rightarrow |a_3 - \rho a_2^2| \le \frac{5}{36}max\left\{1, \frac{5}{16}|\rho|\right\}$$

Case(i): If $|\rho| < \frac{16}{5}$, then $|a_3 - \rho a_2^2| \le \frac{5}{36}$, sharp with the extremal function $f_2(z) = z + \frac{5}{36}z^3 + \frac{25}{1{,}440}z^5 + \ldots$

Case (ii): If $|\rho| \ge \frac{16}{5}$, then $|a_3 - \rho a_2^2| \le \left(\frac{5}{24}\right)^2|\rho|$, this result is sharp with extremal function $f_1(z) = z + \frac{5}{24}z^2 + \frac{1}{216}z^6 + \ldots$

Corollary 1: If $f \in C_s(\varphi_{4L})$ be of the form (1.1), then

$$\left|H_{2,1}(f)\right| = |a_3 - a_2^2| \le \frac{5}{36}.$$

Proof: By taking $\rho = 1$ in the above theorem, we get this result.

Theorem 3.3: If $f \in C_s(\varphi_{4L})$ be of the form (1.1), Then $|T_2(1)| \le \frac{601}{576}$ and $|T_2(2)| \le \frac{325}{5184}$.

Proof: We have $|T_2(1)| \le |1 - a_2^2| \le |1| + |a_2^2| \le \frac{601}{576}$.

Consider $g(z) = z + \left(\frac{5i}{24}\right)z^2 + \left(\frac{i}{216}\right)z^6 + \cdots \in C_s(\varphi_{4L})$.

Here $a_2 = \left(\frac{5i}{24}\right)$ and $|1 - a_2^2| = 1 - \left(\frac{-25}{576}\right) = \frac{601}{576}$. Thus, $g(z)$ is an extremal function for the inequality $|T_2(1)| \le \frac{601}{576}$.

Further, $|T_2(2)| = |a_2^2 - a_3^2| \leq |a_2^2| + |a_3^2| = \left(\frac{5}{24}\right)^2 + \left(\frac{5}{36}\right)^2 = \frac{325}{5184}$.

Theorem 3.4: If $f \in C_s(\varphi_{4L})$ be of the form (1.1), Then

$|a_4 - a_2a_3| \leq \frac{5}{96}$ and the result is sharp.

Proof: As $f \in C_s(\varphi_{4L})$, in view of (3.5),(3.6) and (3.7), we have

$$a_2a_3 - a_4 = \left(\frac{5}{24}\right)^2 \frac{1}{6}\left(c_1c_2 - \frac{1}{2}c_1^3\right) - \left(\frac{5}{24}\right)\frac{1}{8}\left(\frac{7}{48}c_1^3 - \frac{19}{24}c_1c_2 + c_3\right)$$

On simplification and by taking modulus on both sides, we obtain

$$|a_4 - a_2a_3| = \frac{5}{192}\left|\frac{41}{144}c_1^3 - \frac{36}{72}c_1c_2 + c_3\right|$$

By applying the Lemma 2.3, we get

$|a_4 - a_2a_3| \leq 2\left(\frac{5}{192}\right) = \left(\frac{5}{96}\right)$.

This result is sharp for the extremal function is $f_3(z) = z + \frac{5}{96}z^4 + \cdots$

Theorem 3.5: If $f \in C_s(\varphi_{4L})$ be of the form (1.1), Then $|a_5 - a_3^2| \leq \frac{1}{24}$.

This result is sharp.

Proof: In view of equations (3.6) and (3.8), we have

$$a_3^2 - a_5 = \left(\frac{5}{72}\right)^2 \left(c_2 - \frac{1}{2}c_1^2\right)^2 + \frac{1}{48}\left(\frac{7}{96}c_1^4 + \frac{7}{24}c_2^2 + c_1c_3 - \frac{13}{24}c_1^2c_2 - c_4\right)$$

On simplification and by taking modulus on both sides, we obtain,

$$|a_3^2 - a_5| = \frac{1}{48}\left|\frac{113}{864}c_1^4 + \frac{113}{216}c_2^2 + c_1c_3 - \frac{167}{216}c_1^2c_2 - c_4\right|$$

$$= \frac{1}{48}\left|lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3n}{2}c_1^2c_2 - c_4\right|$$

Where $l = \frac{113}{864}, m = \frac{1}{2}, n = \frac{167}{324}, r = \frac{113}{216}$.

It is clear that l, r, m, n satisfy the hypothesis of the Lemma (2.5) and hence $|a_5 - a_3^2| \leq \frac{1}{24}$.

Theorem 3.6: If $f \in C_s(\varphi_{4L})$ be of the form (1.1), Then $|H_{3,1}(\mathrm{f})| \leq \frac{235}{27648}$.

Proof: It is found in [15] that $|H_{3,1}(\mathrm{f})| \leq |a_5 - a_3^2||a_3 - a_2^2| + |a_4 - a_2a_3|^2$.

By utilizing the bounds obtained in the Theorems (3.4), Theorem (3.5) and Corollary (3.1),

we obtain $$|H_{3,1}(f)| \leq \left(\frac{1}{24}\right)\left(\frac{5}{36}\right) + \left(\frac{5}{96}\right)^2 = \frac{235}{27648} \cong 0.0084997$$.

Theorem 3.7: If $f \in C_s(\varphi_{4L})$ be of the form (1.1), Then $|T_3(1)| \leq \frac{2867}{2592}$.

Proof: We have $|T_3(1)| \leq 1 + 2|a_2|^2 + |a_3||a_3 - 2a_2^2|$

From Theorem 3.1, $|a_3 - 2a_2^2| \leq \frac{5}{36} max\left\{1, \frac{5}{16}|2|\right\} = \frac{5}{36}$

$\Rightarrow |T_3(1)| \leq 1 + 2\left(\frac{5}{24}\right)^2 + \left(\frac{5}{36}\right)^2$.

Hence, $|T_3(1)| \leq \frac{2867}{2592} \cong 1.10609568.$

Conclusion:

In this paper, upper bounds of third order Hankel and Toeplitz determinants for functions in the class $C_s(\varphi_{4L})$ are obtained. Also, one can try for upper bounds of higher order determinants for the same class.

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