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EXISTENCE AND UNIQUENESS OF IMPULSIVE STOCHASTIC NEUTRAL PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS DRIVEN BY A FRACTIONAL BROWNIAN MOTION

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ABSTRACT

This article presents the results on existence and uniqueness of mild solution of impulsive stochastic neutral partial functional differential equations driven by a fractional Brownian motion. The results are obtained by using the method of successive approximation and Bihari's inequality.

Keywords

Existence, Uniqueness, Bihari inequality, Fractional Brownian motion, Successive Approximation.

1 Introduction

Stochastic neutral functional differential equations assume a vital role in various sciences such as Physics, Mechanical Engineering, Control Theory and Economics where in, quite often stated that the future state of such systems depends not only on the present state but also on its past history leading to stochastic neutral functional differential equations rather than stochastic differential equations. Several authors [2,15, 18, 27, 70] have probed into the problems of existence of solutions of stochastic differential equations driven by fBm. However the randomness occurs when describing such Phenomena and as such can be used to provide a more accurate mathematical description of the Phenomena. Boufoussi and Hajji [13] obtained the existence results for neutral stochastic functional differential equations driven by fBm in a Hilbert space. Subsequently Caraballo et al., [15] derived an exponential behaviour of solutions to stochastic delay evolution equations with fBm, very recently by Nguyen- Tien- Dung [68] established neutral stochastic differential equations driven by a fBm with impulsive effects and varying time delays. On the other hand neutral stochastic differential equations with infinite delay have become important in recent years as mathematical models of phenomena in both physical and social sciences.

Motivated by the above mentioned work, the main purpose of this paper is to deal with the study of impulsive stochastic neutral partial functional differential equations. Our results are established by using Bihari inequality and successive approximation method. The main work in this paper is to discuss the existence, uniqueness of solutions to impulsive neutral system employing the non Lipschitz condition and Lipschitz condition.

The paper is organised as follows. In section 2 we recall briefly the notations, definitions, lemmas and preliminaries which are used throughout this paper. In section 3, we investigate the existence and uniqueness of impulsive stochastic neutral partial functional differential equations driven by fBm.

In this chapter, we discuss the stochastic neutral functional differential equations of the form

$$D[x(t)+u(t,x_t)] = [Ax(t)+f(t,x_t)]dt + g(t,x)dw(t) + \sigma(t)dB_Q^H(t),$$
$$t \neq t_{k,} \qquad t \in [0,T], \;\; 0 \leq t \leq T$$
$$\Delta x(t_k) = x(t_k{}^+) - x(t_k{}^-) - \mathrm{I_k}\left(x(t_k)\right)$$
$$t = t_{k,}\; k = 1,2,3,\ldots\ldots..m$$
$$x(t) = \varphi \in D_{B_0}^b\left((-\tau,0)\right), x_\xi \qquad (1)$$

where A is the infinitesimal generator of an analytic semi group of bounded linear operators, $\{S(t)\}_{t\geq o}$ with $D \subset X$, B_Q^H is a fBm. $u: [0,\ T\,] \times D_\xi \;\rightarrow\; \mathcal{L}_Q^0(Y,\ \mathrm{X})$; $f,\ g : [0,\ T\,] \times D_\xi \;\rightarrow X,\ \sigma : [0,\ T] \;\rightarrow\; \mathcal{L}_Q^0(Y,\ \mathrm{X})$ are Borel measurable .and $D_\xi \in C\left([-\tau,0]\,;\ \mathcal{L}_Q^0(Y,\ X)\right)$. Here $\mathcal{L}_Q^0(Y,\ \mathrm{X})$ denotes the space of all Q-Hilbert- Schmidt operators from Y into X .

2 Preliminaries

Let $(\Omega,\ \mathcal{F},\mathbb{P})$ be a complete probability space. Consider a time interval $[0,T]$ with arbitrarily fixed horizon T and let $\{\beta^H(t),\ t\ \in\ [0,\ T]\}$ be the one dimensional fBm with Hurst parameter $H \in \left(\frac{1}{2},\ 1\right)$. This implies that by definition β^H is a centered Gaussian process with covariance function:

$$R_H(t,s) = \mathbb{E}(\beta_t^H\,\beta_s^H) \;=\; \frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H}).$$

Further β^H has the resulting Wiener integral representation:

$$\beta^H(t) = \int_0^t \mathrm{K}_H(t,s)d\beta(s)$$

where $\beta = \{\beta^H(t),\ t \in [0,T]\}$ is a Wiener process and $\mathrm{K}_H(t,\ s)$ is the kernel given by

$$\mathrm{K}_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{1}{2}}\, du$$

for $t > s$, where $c_H = \sqrt{H\,(2H-1)/\beta\left(2-2H,\ H-\frac{1}{2}\right)}$ and $\beta(\,\cdot,\cdot)$ represents the Beta function.

We put $\mathrm{K}_H(t,\ s) = 0$ if $t \leq s$ and $\mathcal{H}$ *be* the reproducing kernel Hilbert space of the fBm, and is the closure of set of indicator functions $\{\mathrm{I}_{[0,t]},\ t \in [0,T]\}$ with respect to the scalar product

$$\langle \mathrm{I}_{[0,t]},\ \mathrm{I}_{[0,s]} \rangle_{\mathcal{H}} = R_H\,(t,\ s).$$

The mapping $\mathrm{I}_{[0,t]} \to \beta^H(t)$ can be projected to an isometry between $\mathcal{H}$ and the first Wiener chaos and we will denote by $\beta^H(t)$ the image of φ by the previous isometry.

We recall that for $\psi,\ \varphi \in \mathcal{H}$ their scalar product in $\mathcal{H}$ is given by

$$\langle \psi, \varphi \rangle_{\mathcal{H}} = H\,(2H-1) \int_0^T \int_0^T \psi(s)\ \varphi(t)|t-s|^{2H-2}\, ds\, dt.$$

Let us consider the operator K_H^* from $\mathcal{H}$ to $L^2([0,T])$ which is defined by

$$(K_H^*\ \varphi)(s) = \int_s^T \varphi\,(r)\frac{\partial \mathrm{K}}{\partial r}(r,s)dr$$

Then K_H^* is an isometry between $\mathcal{H}$ and $L^2([0,\ T])$.

Further for any $\varphi \in \mathcal{H}$, we have

$$\beta^H(\varphi) = \int_0^t (K_H^*)(t)\, d\beta(t).$$

It follows from [71] that the elements of $\mathcal{H}$ may be not functions but distributive of negative order. In order to get a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ generated b the measurable functions ψ such that

$$\|\psi\|_{|\mathcal{H}|}^2 = \alpha_H \int_0^T \int_0^T |\psi(s)|\,|\psi(t)|\,|t-s|^{2H-2}\, ds\, dt < \infty,$$

where $\alpha_{\mathcal{H}} = H(2H-1)$. The space $|\mathcal{H}|$ is a Banach space with the norm $\|\psi\|_{|\mathcal{H}|}$.

Definition 2.1

Let $\varphi: [0,T] \to \mathcal{L}_Q^0(\mathbb{K}, \mathbb{H})$ such that

$$\sum_{n=1}^{\infty} \left\| K_H^*(\varphi\ Q^{\frac{1}{2}} e_n) \right\|_{L^2([0,T];\ \mathbb{H})} < \infty. \qquad (2)$$

Then, its stochastic integral with respect to the fBm $B_Q^H(t)$ is defined, for $t \geq 0$, as follows:

$$\int_0^t \varphi(s) dB_Q^H(s) = \sum_{n=1}^{\infty} \int_0^t \varphi(s)\, Q^{\frac{1}{2}} e_n d\beta_n^H$$

$$= \sum_{n=1}^{\infty} \int_0^t \left(K_H^* \left(\varphi\, Q^{\frac{1}{2}} e_n\right)\right)(s)\, d\beta(s).$$

Lemma 2.1 [15]

For any $\varphi: [0, T] \to \mathcal{L}_Q^0(\mathbb{K}, \mathbb{H})$ such that

$$\sum_{n=1}^{\infty} \left\| \varphi\, Q^{\frac{1}{2}} e_n \right\|_{L^{1/H}([0,T];\, \mathbb{H})} < \infty \quad (3)$$

Holds, and for any $\alpha, \beta \in [0, T]$ with $\alpha > \beta$, we have

$$\mathbb{E} \left\| \int_\alpha^\beta \varphi(s) dB_Q^H(s) \right\|^2 \leq cH(2H-1)(\alpha - \beta)^{2H-1} \sum_{n=1}^{\infty} \int_\alpha^\beta \left\| \varphi\, Q^{\frac{1}{2}} e_n \right\|^2 ds,$$

where $c = c(H)$. In addition, if $\sum_{n=1}^{\infty} \left\| \varphi(t) Q^{\frac{1}{2}} e_n \right\|$ is uniformly convergent for $t \in [0, T]$, then

$$\mathbb{E} \left\| \int_\alpha^\beta \varphi(s) dB_Q^H(s) \right\|^2 \leq cH(2H-1)(\alpha - \beta)^{2H-1} \int_\alpha^\beta \|\varphi\|_{\mathcal{L}_Q^0}^2 ds.$$

Lemma 2.1

Suppose that the preceding conditions are satisfied.

(1) Let $0 < \alpha \leq 1$. Then X_α is a Banach space.

(2) If $0 < \beta \leq \alpha$ then the injection $X_\alpha \to X_\beta$ is continuous.

(3) For every $0 < \beta \leq 1$ there exists $M_\beta > 0$ such that

$$\left\|(-A)^\beta S(t)\right\| \leq \frac{M_\beta}{t^{2\beta}}, \quad t > 0, \ \lambda > 0.$$

Lemma 2.2 (Mao [63, Theorem 1.8.2, P.45])

Let $T > 0$ and $c > 0$. Let $\kappa : \mathbb{R}^+$ to $\mathbb{R}^+$ be a continuous nondecreasing function such that $\kappa(t) > 0$ for all t > 0. Let $u(\cdot)$ be a Borel measurable bounded nonnegative function on $[0, T]$. If

$$u(t) \leq c + \int_0^t v(s)\kappa(u(s))ds \quad \text{for all} \quad 0 \leq t \leq T.$$

$$u(t) \leq J^{-1}\left(J(c) + \int_0^t v(s)ds\right)$$

holds for all such $t \in [0, T]$ that

$$J(c) + \int_0^t v(s)ds \in \text{Dom}\,(J^{-1}),$$

where $J(r) = \int_0^r ds/\kappa(s)$ on $r > 0$, and J^{-1} is the inverse function of J.

Definition 2.2

A X-valued process $x(t)$ is called a mild solution of (1.1) if

$$x \in \left([-\tau,\mathrm{T}],\mathbb{L}^2(\Omega,\mathrm{X})\right) \text{ for } \mathrm{t}\in[-\tau,0],\quad x(t)=\varphi(t), \text{ and for } \mathrm{t}\in[0,\mathrm{T}]$$

satisfies

$$\begin{aligned} x(t) = S(t)[\xi(0)+u(0,\xi)] - u(t,x_t) \\ -\int_0^t AS(t-s)u(s,{x_s}^{n-1}ds) \\ +\int_0^t S(t-s)f(s,x_s)ds + \int_0^t S(t-s)g(s,x_s)dw(s) + \int_0^t S(t-s)\,\sigma(s)dB_Q^H(s) \\ + \sum_{k=1}^{m} S\,(t-t_k)\mathrm{I}_{\mathrm{K}}\left(x^{n-1}(t_k)\right) \qquad (4) \end{aligned}$$

3. Existence and Uniqueness

In this section, we study the existence and uniqueness of mild solution for Equation (3.1). For this equation, we assume that the following conditions hold:

(H1) A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ of bounded linear operators on X. We assume that $0 \in \rho(A)$ and

$$\|S(t)\| \leq \mathrm{M} \text{ and } \left\|(-A)^{\beta}\, S(t)\right\| \leq \frac{M_{1-\beta}}{t^{1-\beta}},$$

for some constants $M,\ M_\beta$ and every $t \in [0,\ T]$.

(H2) The function f and h holds good in respect of the following non-Lipschitz condition: for any $x,\ y \in X$ and $t \geq 0$,

$$\|f(t,x)-f(t,y)\|^2 \vee \|g(t,x)-g(t,y)\|^2 \leq \kappa(\|x-y\|^2),$$

where κ is a concave non-decreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that

$$\kappa(0) = 0,\quad \kappa(\mathrm{u}) > 0$$

and $\int_{0^+} du/\kappa(u) = \infty.$

(H3) There exist constants $1/2 < \alpha \leq 1,\ L_g \geq 0$ such that the function g is X_α-valued and satisfies for any $x, y \in D_\xi$ and $t \geq 0$,

$$\left\|(-A)^{-\xi}g(t,x)-(-A)^{-\xi}g(t,y)\right\|^2 \leq L_g\,\|x-y\|^2$$

(H4) The function $\sigma:\ [0,+\infty) \to \mathcal{L}_2^0\,(Y,\ X)$ satisfies

$$\int_0^T \|\sigma(s)\|_{\mathcal{L}_2^0}^2\, ds < L, \qquad \forall\ \mathrm{T} > 0.$$

(H5) The function $\mathrm{I}_K \in \mathbb{C}\,(\mathbb{y},\mathbb{x})$ and there exists some constants h_k such that

$\|\mathrm{I}_K(x)-\mathrm{I}_K(y)\|^2 \leq h_k\|x-y\|^2$, for $x, y \in \mathbb{H}$ and k = 1,2,3,…..m.

By successive approximation procedure, for each integer n = 1, 2, 3, ……,

$x^n(t) = \mathrm{S(t)}[\xi(0) + u(0,\varphi)] - u(t, x_t^n) - \int_0^t AS(t-s)f(s, x_s^n ds) + \int_0^t S(t-s)f(s,x)ds + \int_0^t S(t-s)g(s, x_s^{n-1})dw(s) + \int_0^t T(t-s)\sigma(s)\, dB_Q^H(s) \quad +\sum_{k=1}^m S(t - t_k)\mathrm{I_K}\big(x^{n-1}(t_k)\big)$

and for n = 0,

$$x^0(t) = S(t)\xi(0), \qquad t \in [0,\ T].$$

While for n = 1, 2,……..

$$x^n(t) = \xi(t), \qquad t \in [-\tau, 0]. \qquad (5)$$

Theorem 1

Let the assumption (H1) - (H6) hold. Then the system (1) has unique mild solution $x(t)$ in $\mathcal{B}_T$

$$\mathbb{E}\left\{\sup_{0 \le t \le T} \|x^n(t) - x(t)\|^2\right\} \to 0 \; as\; n \to \infty$$

$where\{x^n(t)\}_{n\ge1}$ are the successive approximations (3)

$$\mathbb{E}\,\|x^n(t)\|^2 \le 7M^2\, \mathbb{E}\,\|\varphi(0) + u(0,\varphi)\|^2 + 14\varphi\left\|-A^{\xi}\right\|^2$$

$$\mathbb{E}\left\|(-A^{\xi})u\,(t, x_t^n) - (-A^{\xi})u(t,0)\right\|^2 + \left\|(-A^{\xi})u\,(t, x_t^n)\right\|^2$$

$$+14T\mathbb{E}\,\int_0^t \left\|-A^{1-\xi}\, S\,(t-s)\right\|^2 \left[\left\|(-A^{\xi})u\,(t, x_s^n) - (-A^{\xi})u\,(s,0)\right\|^2\right] ds$$

$$+\,14M^2T\mathbb{E}\,\int_0^t [\|f(s, x_s^{n-1}) - f(s,0)\|^2 + \|f(s,0)\|^2 ds]$$

$$+7M^2T\mathbb{E}\,\int_0^t [\|g(s, x_s^{n-1}) - g(s,0)\|^2 + \|g(s,0)\|^2 ds]$$

$$+\,14M^2 m\,\mathbb{E}\sum_{k=1}^m\left[\left\|I_K\left(x^{n-1}(t_k) - I_k(0)\right)\right\|^2 + \|I_K(0)\|^2\right].$$

From the lemma 2,3 (H3) and (H6) the following relation holds:

$$\mathbb{E}\|(-A)S(t-s)u\,(s, x_s^n)\|^2 = \mathbb{E}\left\|(-A^{1-\xi})S(t-s)(-A^{\xi})u\,(s, x_s^n)\right\|^2$$

$$\le 2\left[\left\|(-A^{1-\xi})S(t-s)^2\left\|(-A^{\xi})u\,(s, x_s^n) - (-A^{\xi})u(s,0)\right\|E\left\|(-A^{\xi})u\,(s,0)\right\|^2\right\|^2\right]$$

$$\le \frac{2M_{1-\xi}}{(t-s)2^{(1-\xi)}}\left[L_g\mathbb{E}\,\|x^n\|^2{}_s + \kappa_0\right].$$

Thus from the above,

$$\mathbb{E}\,\|x^n(t)\|^2 \le 14M^2[\mathbb{E}\|\varphi(0)\|^2 + \mathbb{E}\|u(0,\varphi)\|^2\,]$$

$$+14\left\|-A^{-\xi}\right\|^2\left[L_g\mathbb{E}\,\|x^n\|^2{}_s + \kappa_0\right]$$

$$+\frac{2M_{1-\xi}}{(t-s)^{2(1-\xi)}}\left[L_g\mathbb{E}\,\|x^n\|^2{}_s + \kappa_0\right]\sum_{k=1}^m\left[I_k\mathbb{E}\,\|x^{n-1}\|^2{}_t + \kappa_0\right]$$

$$+14M^2(T+1)\,\mathbb{E}\int_0^t\left[k\left(\|x^{n-1}\|^2{}_s\right) + \kappa_0\right]\mathrm{ds}$$

$$+7M^2cH(2H-1)T^{2H}L + 14M^2m\sum_{k=1}^{m}\left[I_k\mathbb{E}\,\|x^{n-1}\|^2_{\ t} + \kappa_0\right]$$

$$\mathbb{E}\,\|x^n(t)\|^2_{\ t} \le Q_{2+}\frac{14M^2(T+1)}{1-Q_1}\,\mathbb{E}\int_0^t K\left(\|x^{n-1}\|^2_{\ s}\right)ds$$

$$+\frac{14M^2m\,\mathbb{E}\sum_{k=1}^{m}h_k}{1-Q_1}\,\{\|x^{n-1}\|^2_{\ t}\}.$$

Where $Q_1 = 14\left[\|-A^{-\xi}\|^2 + 2\frac{2M_{1-\xi}}{1-\xi}\,T^{2\xi}\right]L_g$

Given that k(.) is concave and k(0) =0, we can find positive constants a and b such that

$$K(u) \le a + bu, \quad \text{for all} \quad u \ge 0.$$

And applying Mathematical induction, we get

$$\mathbb{E}\,\|x^n(t)\|^2_{\ t} \le Q_{3+}\frac{14M^2(T+1)}{1-Q_1}\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}\,\mathbb{E}\|x^0\|^3_s ds$$

$$+\frac{14M^2m}{1-Q_1}\sum_{k=1}^{m}h_k\left\{\frac{T^{n-1}}{(n-1)!}\,\mathbb{E}\|x^0\|^2_{\ t}\right\} \qquad (6)$$

Where $Q_3 = Q_2 + \frac{14M^2(T+1)a}{1-Q_1}$. We know that,

$$\mathbb{E}\,\|x^0(t)\|^2 \le M^2\mathbb{E}\,\|\varphi(0)\|^2 = Q_4 < \infty. \qquad (7)$$

Thus $\mathbb{E}\|x^0(t)\|^2_t \le \infty$, then from ( ),

$$\sup_{0\le t\le T}\ \mathbb{E}\|x^0(t)\|^2_t \le \infty, \quad \text{for all n= 1, 2, 3,……} \qquad (8)$$

This proves the boundedness of $\{x^n(t)\}$ and $t \in [0,T]$

The sequence { x^n (t) } is a Cauchy sequence.

Let us next show that { x^n (t) } is a Cauchy sequence in T. For this, choose Ti ϵ [0,T) such that

$\frac{4M^2T_1}{1-Q_5}\,K\,\frac{(t-a)^n}{n!}\,Q_9 \le \frac{4M^2T_i}{1-Q_5}\frac{(t-s)}{n!}\,Q_9$, for all $0 \le t \le T_1$,

Consider,

$$E\,\|x^{n+1}(t) - x^n(t)\|^2 \le 4\left[\,\|-A^{-\epsilon}\|^2 + \frac{c_1-\epsilon}{2\epsilon-1}T_1^{2\epsilon}\,\right]L_gE\,\|x^{n+1}(t) - x^n(t)\|^2_t$$

$$+\,4M^2T_1\int_0^t K\,(E\,\|x^n - x^{n-1}\|^2_t\,ds$$

$$4M^2m\sum_{k=1}^{m}h_k\,E\,\|x^n - x^{n-1}\|^2_t$$

Thus,

$$E\,\|x^{n+1}(t) - x^n(t)\|^2_t \le \frac{4M^2T_1}{1-Q_5}\int_0^t K\,(E\,\|x^n - x^{n-1}\|^2_t\,ds + \frac{4M^2m\sum_{k=1}^{m}h_k}{1-Q_5}\;E\,\|x^n - x^{n-1}\|^2_t \qquad (9)$$

Where, $Q_5 = 4[\,\|-A^{-\epsilon}\|^2 + \frac{C_{1-\epsilon}}{2\epsilon-1}T_1^{2\epsilon}]$ Lg. Moreover,

$$\|x^1(t) - x^0(t)\|^2 = \Big\|S\,(t)u\,(0,\emptyset(0)) - [u(t,x_t^1\,) - u(t,x_t^0\,)\,] - u\,(t,x_t^0\,) - \int_0^t AS(t-s)[\,u(S,x_s^0)ds + \int_0^t AS(t-s)[\,h(S,x_s^0)ds + \int_0^t s(t-s)f(s,x_s^0\;ds + \int_0^t s(t-s)\sigma(s)dB^H(s) + \sum_{0<t_k<t}s(t-t_{k)}\;I_k\;(x^0(t_{k)})\Big\|^2$$

Then we get

$$\|x^1(t) - x^0(t)\|^2{}_t \le Q_6 + \frac{16\left(\left[\|-A^{-\epsilon}\|^2 + \frac{c_{1-\epsilon}}{2\epsilon-1}\right]L_g + M^2 m \sum_{k=1}^{m} h_k\right)}{1-Q_7} \mathrm{E}\,\|x^0\|_t^2 + \frac{16M^2T_1}{1-Q_7}\int_0^t K(E\|x^0\|_s^2)ds,$$

Taking supremum over t, and from (10), weget

$$\underset{t\in[0,T_1]E}{Sup}\ \|x^1 - x^0\|_t^2 \le Q_8 + \frac{16M^2T_1}{1-Q_7}\int_0^t K\,(Q_4)\,ds$$

Thus applying mathematical induction in (9) and from (10),

$$\underset{t\in[0,T_1]}{Sup}\ E\|x^{n+1}(t) - x^n(t)\|_t^2 \le \frac{4M^2T_1}{1-Q_5}\int_0^t K\,\frac{(t-s)^n}{n!}\ \underset{t\in[0,T_1]}{Sup}\ E\|x^1 - x^0\|_s^2\,\mathrm{ds}$$

$$+\frac{4M^2\mathrm{m}\sum_{k=1}^{m} h_k}{1-Q_5}\left\{\frac{T_1^n}{n!}\right\}\underset{t\in[0,T_1]}{Sup}\ E\|x^1 - x^0\|_t^2$$

$$\le \frac{4M^2T_1}{1-Q_5}\int_0^t K\left(\frac{(t-s)^n}{n!}\ Q_9\right)\mathrm{ds} + \frac{4M^2\mathrm{m}\sum_{k=1}^{m} h_k}{1-Q_5}\left\{\frac{T_1^n}{n!}\right\} Q_9$$

$$\le \frac{4M^2T_1}{1-Q_5}\int_0^t \frac{(t-s)^n}{n!}\ Q_9\mathrm{ds} + \frac{4M^2\mathrm{m}\sum_{k=1}^{m} h_k}{1-Q_5}\left\{\frac{T_1^n}{n!}\right\} Q_9$$

$$\le Q_{10}\,\frac{T_1^n}{n!}, \mathrm{n} \ge 0,\ \mathrm{t} \in [0, T_1]$$

Note that for any m > n ≥ 1, we have

$$\underset{t\in[0,T_1]}{Sup}\ E\|x^{n+1}(t) - x^n(t)\|_t^2 \le \sum_{r=n}^{\infty}\ \underset{t\in[0,T_1]}{Sup}\ E\|x^{r+1} - x^r\|_t^2$$

$$\le \sum_{r=n}^{\infty}\left(Q_{10}\frac{T_1^r}{r!}\right)$$

$$\to 0 \text{ as } n \to \infty \qquad (11)$$

This shows that $\{x^n\}$ is Cauchy in B_T

Then the standard Borel -cantelli lemma argument can be used to show that $x^n(t) \to x(t)$ uniformly in t on $[0, T_1]$hence $x(t)$ is a sollutions of (1) in the interval $[0, T_1]$. By theorem 6 in [2], the existence of solution of (1) on $[0, T]$ can be obtained by iteration.

Now, we prove the uniqueness of the solution (2). Let $x_1, x_2 \in \mathcal{B}_T$ be two solutions to (1) on some interval $(-\tau, T)$ then for $t \in (-\tau, 0)$, the uniqueness is obvious for $(0 \le t \le T)$, we have

$$\mathbb{E}\|x_1(t) - x_1(t)\|^2 \le 5\left(\left[\left\|1 - A^{-\xi}\right\|^2 + \frac{M_{1-\xi}}{2\xi-1}\right]L_g + +M^2 m\ \sum_{k=1}^{m} h_k\right)\ \mathbb{E}\|x_1 - x_1\|^2{}_t$$

$$+\ 5M^2(T+1)\int_0^t k\left(\mathbb{E}\|x_1 - x_1\|^2{}_s\right)ds$$

Thus

$$\mathbb{E}\|x_1(t) - x_2\,(t)\|^2{}_t \le \frac{5M^2(T+1)}{1-Q_{11}}\int_0^t k\left(\mathbb{E}\|x_1 - x_1\|^2{}_s\right)ds$$

Where $Q_{11} = 5\left(\left[\left\|1 - A^{-\xi}\right\|^2 + \frac{M_{1-\xi}}{2\xi-1}\right]L_g + +M^2 m\ \sum_{k=1}^{m} h_k\right)$.

Thus, Bihari inequality yields that

$$\underset{t\,\in\,[0,T]}{sup}\ \mathbb{E}\|x_1 - x_2\,\|^2{}_t = 0,\quad 0 \le t \le T$$

Thus, $x_1(t) = x_2\,(t)$, for all $-\infty < t \le T$,

There fore, for ll $-\infty < t \le T$,

$x_1(t) = x_2\,(t)$.

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