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Quadratic Functional Differential Equation

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Abstract: In this paper, we discuss the quadratic functional differential equation on unbounded intervals for existence as well as for uniformly global attractivity of the solution. We apply classical hybrid fixed point theory to obtain the results.

Keywords: Functional differential equation; Fixed point theorem, uniformly global attractivity, Quadratic functional differential equation.

1. Introduction

Consider the following quadratic functional differential equation on unbounded intervals,,

$$\begin{aligned}\left[\frac{a(t)x(t)}{f(t,x(t))}\right]' &= g(t,x(t),x_t)+h(t,x(t),x_t) \quad a.e.\, t\in R_+ \\ x_0 &= \phi\end{aligned} \tag{1}$$

Where $a \in CRB(R_+), f: R_+ \times R \to R\setminus\{0\}$, $g: R_+ \times R \times C \to R$ and $h: R_+ \times R \times C \to R.$

The quadratic functional differential equation (1) is new to the theory of nonlinear differential equations and some special cases of these quadratic functional differential equation with $a = 1$ have been studied in the literature on closed and bounded intervals for various aspects of the solutions Hale [13] , Ntouyas [16] Dhage [11].The QFDE (1)is not discussed on closed but unbounded intervals of real line. In this paper,we discuss the quadratic perturbations of the first order ordinary differential equation for existence as well as for different characterizations of the solutions such as attractivity, asymptotic attractivity and ultimate positivity of the solutions using hybrid fixed point theory.

2. Existence Results

Let X be a non-empty set and let T : $X \to X$. An invariant point under T in X is called a fixed point of T, that is, the fixed points are the solutions of the functional equation T $x = x$. Any statement asserting the existence of fixed point of the mapping T is called fixed point theorem for the mapping T in X. We give some fixed point theorems useful in the attrratctivity and ultimate positivity of the solutions for functional differential equation (1) on unbounded intervals.

Theorem 2.1 (Granas and Dugundji) [12]. Let S be a non-empty, closed, convex and bounded subset of the Banach space X and let $Q : S \to S$ be a continuous and compact operator. Then the operator equation $Qx = x$ has a solution in S.

The following fixed point theorem of Burton [3] which is a special case of a hybrid fixed point theorem [11] of Banach spaces.

Theorem2.2 (Dhage[7]). Let S be a closed, convex and bounded subset of the Banach space X and let $A: X \to X$ *and* $B: S \to X$ be two operator such that

(i) A is nonlinear D-contraction,

(ii) B is completely continuous,

(iii) $x = Ax + By \Rightarrow x \in S$ *for all* $y \in S$.

Then the operator equation $Ax + Bx = x$ has a solution in S.

Theorem2.3 (Dhage[10]). Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and Let $A: X \to X$ *and* $B: S \to X$ be two operators such that

(i) A is D-Lipschitz with D-function ψ,

(ii) B is completely continuous,

(iii) $x = Ax\,By \Rightarrow x \in S$ *for all* $y \in S$, *and* $M\psi(t) < r$, *where* $M = \|B(S)\| = \sup\{\|Bx\| : x \in S\}$

Then the operator equation $Ax\,Bx = x$ has a solution in S.

3. Characterizations of Solutions

We find the solutions of the FDE (1) in the space $BC(I_0 \cup R_+, R)$ of continuous and bounded real-valued functions defined on $I_0 \cup R_+$. Define a standard supremum norm $\|\,.\|$ and a multiplication " . " in BC $(I_0 \cup R_+, R)$ by

$$\|x\| = \sup_{t \in I_0 \cup R_+} |x(t)| \text{ and } (xy)(t) = x(t)y(t), t \in R_+$$

Clearly, BC $(I_0 \cup R_+, R)$ becomes a Banach algebra with respect to the above norm and the multiplication in it. By $L^1(R_+, R)$ we denote the space of lebesgue integrable functions on R_+ and the norm $\|\cdot\|_{L^1}$ *in* $L^1(R_+, R)$ is defined by

$$\|x\|_{L^1} = \int_0^\infty |x(t)|\, ds.$$

We assume that $E = BC(I_0 \cup R_+, R)$ and let Ω be a non-empty subset of X. Let Q : $E \to E$ be a operator and consider the following operator equation in E, $Qx(t) = x(t)$ for all $t \in I_0 \cup R_+$.

We give different characterizations of the solutions for the operator equation $Qx(t) = x(t)$ in the space $BC(I_0 \cup R_+, R)$.

Definition 3.1. We say that solutions of the operator equation $Qx(t) = x(t)$ are locally attractive if there exists a closed ball $\bar{B}_r(x_0)$ in the space $BC(I_0 \cup R_+, R)$ for some $x_0 \in BC(I_0 \cup R_+, R)$ such that for arbitrary solutions $x = x(t)$ *and* $y = y(t)$ of equation $Qx(t) = x(t)$ belonging to $\bar{B}_r(x_0)$ we have that

$$\lim_{t \to \infty} (x(t) - y(t)) = 0$$

In the case when the limit ,is uniform with respect to the set $\bar{B}_r(x_0)$, i.e., when for each ∈> 0 there exists T >0 such that

$$|x(t)-y(t))\leq \in$$

for all x, y ∈ $\bar{B}_r(x_0)$ being solutions of $Qx(t)=x(t)$ and for $t \geq T$, we will say that solutions of equation $Qx(t)=x(t)$ are uniformly locally attractive on $I_0 \cup R_+$.

Definition 3.2. A solution $x = x(t)$ of equation $Qx(t)=x(t)$ is said to be globally attractive if $\lim_{t\to\infty}(x(t)-y(t))=0$ holds for each solution $y = y(t)$ of $Qx(t)=x(t)$ in $BC(I_0 \cup R_+, R)$. In other words, we may say that solutions of the equation $Qx(t)=x(t)$ are globally attractive if for arbitrary solutions x(t) and y(t) of $Qx(t)=x(t)$ in $BC(I_0 \cup R_+, R)$ The condition $\lim_{t\to\infty}(x(t)-y(t))=0$ is satisfied. In the case when the condition $\lim_{t\to\infty}(x(t)-y(t))=0$ is satisfied uniformly with respect to the space $BC(I_0 \cup R_+, R)$ i.e., if for every ∈>0 there exists T>0 such that the inequality $\lim_{t\to\infty}(x(t)-y(t))=0$ is satisfied for al x,y,∈ $BC(I_0 \cup R_+, R)$ being the solutions of $Qx(t)=x(t)$ and for $i \geq T$, we will say that solutions of the equation $Qx(t)=x(t)$ are uniformly globally attractive on $I_0 \cup R_+$.

We introduce the new concept of local and global ultimate positivity of the solutions for the operator equation $Qx(t)=x(t)$ in the space $BC(I_0 \cup R_+, R)$.

Definition 3.3 . A solution x of the equation $Qx(t)=x(t)$ is called locally ultimately positive if there exists a closed ball $\bar{B}_r(x_0)$ in the space $BC(I_0 \cup R_+, R)$ for some x_0 ∈ $BC(I_0 \cup R_+, R)$ such that x ∈ $\bar{B}_r(x_0)$ and

$$\lim_{t\to\infty}\left[|x(t)|-x(t)\right]=0.$$

In the case when this limit, is uniform with respect to the solution set of the operator equation $Qx(t)=x(t)$ in $BC(I_0 \cup R_+, R)$ i.e., when for each ∈ > 0 there exists T > 0 such that

$$\left\|x(t)\right| x(t)-x(t)\right|\leq \in$$

For all x being solutions of $Qx(t)=x(t)$ in $BC(I_0 \cup R_+, R)$ and for $t \geq T$, we will say that solutions of equation $Qx(t)=x(t)$ are uniformly locally ultimately positive on R_+.

Definition 3.4 .A solution x ∈ $BC(I_0 \cup R_+, R)$ of the equation $Qx(t)=x(t)$ is called globally ultimately positive if $\lim_{t\to\infty}\left[|x(t)|-x(t)\right]=0.$ is satisfied. In the case when the limit $\left\|x(t)\right| x(t)-x(t)\right|\leq \in$ is uniform with respect to the solution set of the operator equation $Qx(t)=x(t)$ in $BC(I_0 \cup R_+, R)$ i.e., when for each ∈ > 0 there exists T > 0 such that limit, is satisfied for all x being solutions of $Qx(t)=x(t)$ in $BC(I_0 \cup R_+, R)$.

and for $t \geq T$, we will say that solutions of equation $Qx(t)=x(t)$ are uniformly globally ultimately positive on $I_0 \cup R_+$.

Main Result

We prove the global attractivity and positivity results for the functional differential equation(1) on $I_0 \cup R_+$ under some suitable conditions. Let I be a closed interval in R and let $AC(I,R)$ be the space of functions which are defined and absolutely continuous on I.

First, we prove the global attractivity and ultimate positivity results for the functional differential equation (1) on $I_0 \cup R_+$.

Definition3.5. By a solution for the functional differential equation (1) we mean a function $x \in BC(I_0 \cup R_+, R) \cap AC(R_+, R)$ *such that*

the function $t \mapsto \dfrac{a(t)x(t)}{f(t,x(t))}$ is absolutely continuous on R_+ and

x satisfies the equations in (1) on $I_0 \cup R_+$.

where $AC(R_+, R)$ is the space of absolutely continuous real-valued functions on right half real axis R_+.

Consider the following set of hypotheses.

(A_1). There exists a continuous function h : $R_+ \to R_+$ such that $|g(t,x,y)| \le h(t)$ $a.e.\ t \in R_+$

for all $x \in R$ and $y \in C$. Moreover, we assume that $\lim_{t\to\infty} |\bar{a}(t)| \int_0^t h(s)ds = 0$

$(A_2)\ \phi(0) \ge 0$.

(A_3). The function $t \to f(t,0,0)$ is bounded on R_+ with $F_0 = \sup\{|f(t,0,0)| : t \in R_+\}$.

(A_4) .The function $f : R_+ \times R \to R$ is continuous and there exists a function $\ell \in BC(R_+, R)$ and a real number K > 0 such that

$$|f(t,x) - f(t,y)|\, \ell(t) \frac{|x-y|}{K + |x-y|}$$

for all $t \in R_+$ *and* $x, y \in R$ Moreover, we assume $\sup_{t\ge 0} \ell(t) = L.$

(A_5). $\lim_{t\to\infty} [|f(t,x)| - f(t,x)] = 0$ *for all* $x \in R.$

(A_6) $f(0,\phi(0)) \ge 0.$

(A_7) . $f(0,\phi(0)) = 1$

(A_8). The function $x \mapsto \dfrac{x}{f(0,x)}$ *is injective in* R_+.

Theorem 3.1. Assume that the hypotheses (A_1), (A_3), (A_4), (A_7) and(A_8)) hold. Further, assume that

$$L \max\{\|\phi\|, |\phi(0)|\,\|\bar{a}\| + W\} \le K. \qquad (3)$$

Then the functional differential equation (1) has a solution and solutions are uniformly globally attractive on $I_0 \cup R_+$.

Proof. Now, using hypotheses (A_7) and $(A_8$) it can be shown that the FDE (1) is equivalent to the functional integral equation

$$x(t)=\begin{cases}[f(t,x(t))]\left(\phi(0)\bar{a}(t)+\bar{a}(t)\int_0^t g(s,x(s),x_s)ds\right), & if\ t\in R_+\\ \phi(t), & if\ t\in I_0\end{cases} \quad (4)$$

Set $X=BC(I_0\cup R_+,R)$ and define a closed ball $\bar{B}_r(0)$ in X centered at origin of radius r given by

$r=\max\{1,L+F_0\}\max\{\|\phi\|,|\phi(0)|\,\|\bar{a}\|+W\}$

Define the operators A on X and B on $\bar{B}_r(0)$ by

$$Ax(t)=\begin{cases}f(t,x(t)), & if\ t\in R_+\\ 1, & if\ t\in I_0\end{cases} \quad (5)$$

And $$Bx(t)=\begin{cases}\phi(0)\bar{a}(t)+\bar{a}(t)\int_0^t[g(s,x(s),x_s)+h(s,x(s),x_s)]ds, & if\ t\in R_+\\ \phi(t), & if\ t\in I_0.\end{cases} \quad (6)$$

Then the FIE (4) is transformed into the operator equation as

$$Ax(t)\,Bx(t)=x(t),\ t\in I_0\cup R_+. \quad (7)$$

We Show that A and B satisfy all the conditions of Theorem 2.3 on $BC(I_0\cup R_+,R)$ First we show that the operators A and B define the mappings $A:X\to X\ and\ B:\bar{B}_r(0)\to X.$ be arbitrary. Obviously, Ax is a continuous function on $I_0\cup R_+$. We show that Ax is bounded on $I_0\cup R_+$. Thus, if $t\in R_+$, then we obtain:

$$\begin{aligned}|Ax(t)|=|f(t,x(t))|&\le|f(t,x(t))-f(t,0)|+|f(t,0)|\\ &\le\ell(t)\frac{|x(t)|}{K+|x(t)|}+F_0\le L+F_0\end{aligned}$$

Similarly, $|Ax(t)|\le 1$ $for\ all\ t\in I_0$. Therefore, taking the supremum over t,

$\|Ax\|\le\max\{1,L+F_0\}=N$

Thus Ax is continuous and bounded on $I_0\cup R_+$. As a result $Ax\in X$. It can be shown that $Bx\in X$ and in particular, $A:X\to X$ and $B:\bar{B}_r(0)\to X$. We show that A is a Lipschitz on X. Let $x,y\in X$ be arbitrary. Then, by hypothesis (A_3),

$$\begin{aligned}\|Ax-Ay\|&=\sup_{t\in I_0\cup\mathbb{R}_+}|Ax(t)-Ay(t)|\\ &\le\max\left\{\sup_{t\in I_0}|Ax(t)-Ay(t)|,\sup_{t\in\mathbb{R}_+}|Ax(t)-Ay(t)|\right\}\\ &\le\max\left\{0,\sup_{t\in\mathbb{R}_+}\ell(t)\frac{|x(t)-y(t)|}{K+|x(t)-y(t)|}\right\}\\ &\le\frac{L\|x-y\|}{K+\|x-y\|}\end{aligned}$$

for all $x,y\in X$. This shows that A is a D-Lipschitz on X with D-function $\psi(r)=\dfrac{Lr}{K+r}$ next, it can be shown that B is a compact and continuous operator on X and in particular on $\bar{B}_r(0)$ Next , we estimate the value of the constant M. By efinition of M, one has

$$\| B(\bar{B}_r(0)) \| = \sup\{\| Bx \| : x \in \bar{B}_r(0)\}$$

$$= \sup\left\{ \sup_{t \in I_0 \cup R_+} | Bx(t) | : x \in \bar{B}_r(0) \right\}$$

$$\leq \sup\left\{ \max\left\{ \sup_{t \in I_0} | Bx(t) |, \sup_{t \in R_+} | Bx(t) | \right\} : x \in \bar{B}_r(0) \right\}$$

$$\leq \sup_{x \in \bar{B}_r(0)} \{\max\{\| \phi \|, | \phi(0) \| \bar{a}(t) |$$

$$+ \sup_{t \in R_+} | \bar{a}(t) | \int_0^t | g(s, x(s), x_s) + h(s, x(s), x_s) | ds\}\}$$

$$\leq \max \{\| \phi \|, | \phi(0)) \| \bar{a} \| + W\}$$

Thus,

$$\| Bx \| \leq \max\{\| \phi \|, | \phi(0) | \| \bar{a} \| + W\} = M$$

for all $x \in \bar{B}_r(0)$. Next , let $x, y \in X$ be arbitrary. Then,

$$| x(t) | \leq | Ax(t) | \, | By(t) |$$

$$\leq \| Ax \| \, \| By \|$$

$$\leq \| A(X) \| \, \| B(\bar{B}_r(0)) \|$$

$$\leq \max\{1, L + F_0\} M$$

$$\leq \max\{1, L + F_0\} \max\{\| \phi \|, | \phi(0) | \| \bar{a} \| + W\}$$

$$= r$$

For all $t \in I_0 \cup R_+$. Therefore, we have:

$$\| x \| \leq \max\{1, L + F_0\} \max\{\| \phi \|, | \phi(0) | \, \| \bar{a} \| + W\} = r$$

This shows that $x \in \bar{B}_r(0)$ and hypothesis (c) of Theorem 2.3 is satisfied. Again,

$$M \phi(r) \leq \frac{L \max\{\| \phi \| \, | \phi(0) | \, \| \bar{a} \| + W\} r}{K + r} < r$$

For r>0, because

$$L \max\{\| \phi \| \, | \phi(0) | \, \| \bar{a} \| + W\} \leq K.$$

Therefore, hypothesis (d) of Theorem 2.3 is satisfied. Now we apply Theorem 2.3 to the operator equation $Ax\,Bx = x$ to yield that the FDE (1) has a solution on $I_0 \cup R_+$ Moreover, the solutions of the FDE (1) are in $\bar{B}_r(0)$ Hence, solutions are global in nature.

Finally, let $x, y \in \bar{B}_r(0)$ be any two solutions of the FDE (1) on $I_0 \cup R_+$. Then

$$| x(t) - y(t) | \leq | [f(t, x(t))] \left(\phi(0)\bar{a}(t) + \bar{a}(t) \int_0^t [g(s, x(s), x_s) + h(s, x(s), x_s)] ds \right)$$

$$- [f(t, y(t))] \left(\phi(0)\bar{a}(t) + \bar{a}(t) \int_0^t [g(s, y(s), y_s) + h(s, y(s), y_s)] ds \right)$$

$$\leq | f(t, x(t)) - f(t, y(t)) | \left(\phi(0)\bar{a}(t) + \bar{a}(t) \int_0^t [g(s, x(s), x_s) + h(s, x(s), x_s)] ds \right)$$

$$+ | f(t, y(t)) \left(\bar{a}(t) \int_0^t \{[g(s, x(s), x_s) - g(s, y(s), y_s)] - [h(s, x(s), x_s) - h(s, y(s), y_s)]\} ds \right)$$

$$\leq | f(t,x(t)) - f(t,y(t))| \left(|\phi(0)| |\bar{a}(t)| + |\bar{a}(t)| \int_0^t h(s)\,ds \right)$$
$$+2\left[| f(t,x(t)) - f(t,0)| + | f(t,0)| \right] w(t)$$
$$\leq \ell(t) \frac{|x(t)-y(t)|}{K+|x(t)-y(t)|} (|\phi(0)| \, \|\bar{a}\| + W)$$
$$+2\left[\frac{\ell(t)|y(t)|}{K+|y(t)|} + F_0 \right] w(t)$$
$$\leq \frac{L(|\phi(0)| \, \|\bar{a}\| + W)\,|x(t)-y(t)|}{K+|x(t)-y(t)|} + 2(L+F_0)\,w(t) \qquad (8)$$

Taking the limit superior as $t \to \infty$ in the above inequality yields,

$$\lim_{t\to\infty} |x(t) - y(t)| = 0$$

Therefore, there is a real number T > 0 such that $|x(t) - y(t)| < \in for\ all\ t \geq T$. Consequently, the solutions of FDE (1) are uniformly globally attractive on $I_0 \cup R_+$. This completes the proof.

Theorem 3.2. Assume that the hypotheses $(A_1) - (A_6)$ hold. Then the functional differential equation (1) has a solution and solutions are uniformly globally attractive and ultimately positive defined on $I_0 \cup R_+$.

Proof. By Theorem 4.1 the FDE (1) has a global solution in the closed ball $\bar{B}_r(0)$ where the radius r is given as in the proof of Theorem 4.1, and the solutions are uniformly globally attractive on $I_0 \cup R_+$. We know that for any $x, y \in R$, one has the inequality,

$$|x||y| = |xy| \geq xy,$$

and therefore,

$$\| xy| - (xy)| \leq |x| \, \| y| - y| + \| x| - x| \, |y| \qquad (9)$$

for all $x, y, \in R$. Now for any solution $x \in \bar{B}_r(0)$ on has

$$\| x(t)| - x(t)| = \| [f(t,x(t))] \left(\phi(0)\bar{a}(t) + \bar{a}(t) \int_0^t [g(s,x(s),x_s) + h(s,x(s),x_s)]\,ds \right)|$$
$$- ([f(t,x(t))] \left(\phi(0)\bar{a}(t) + \bar{a}(t) \int_0^t [g(s,x(s),x_s) + h(s,x(s),x_s)]\,ds \right)$$

Taking the limit superior as $t \to \infty$ in the above inequality , we obtain $\lim_{t\to\infty} \| x(t)| - x(t)| = 0$ Therefore, there is a real number T > 0 such that $\left| |x(t)| - x(t) \right| \leq \in for\ all\ t \geq T$. Hence, Solutions of the FDE (1) are uniformly globally attractive as well as ultimately positive defined on $I_0 \cup R_+$. This completed the proof.

References

[1] J. Banas, B. Rzepka, An application of a measure of non-compactness in the study of asymptotic stability, Appl. Math. Letter 16(2003),1-6.

[2] J. Banas, B.C. Dhage, Global asymptotic stability of solutions of a functional integral equeations, Nonlinear Analysis 69 (2008), 1945-1952.

[3] T.A. Burton, A fixed point theorem of Krasnoselskii, Appl.Math. Lett. 11(1998),85-88.

[4] T.A. Burton, B. Zhagng, Fixed points and stability of an integral equations: nonuniqueness, Appl. Math. Letters 17(2004), 839-846.

[5] T.A. Burton and T. Furumochi, A note on stability by Schauder's theorem, Funkcialaji Ekvacioj 445(2001), 73-82.

[6] K. Deimling, Nonlinear Functional Analysis, Springer Verlag, Berlin, 1985.

[7] B.C. Dhage, A nonlinear alternative with applications to nonlinear perturbed differential equiations, Nonlinear Studies, 13(4) (2006), 343-354.

[8] B.C. Dhage, Local asymptotic attractivity for nonlinear quadratic functional integral equation, Nonlinear Analysis 70 (5) (2009), 1912-1922.

[9] B.C. Dhage, Global attractivity result for nonlinear functional integral equations via a Krasnoselskii type fixed point theorem, Nonlinear Analysis 70 (2009), 2485-2493

[10] B.C. Dhage, A fixed point theorem in Banach algebras withapplications to functional integral equations, Kyungpook Math. J. 44(2004),145-155.

[11] B.C. Dhage, S.N. Salunkhe, R.P. Agrawal and W. Zhang, A functional differential equations in Banach algebras, Math.Ineq. Appl.8 (1) (2005),89-99.

[12] A.Granas and J. Dugundji, Fixed Point Theory, Springer Verlag,New York, 2003.

[13] H.K. Hale, Theory of Functional Differential Equations, Springer Verlag,New York, 1977.

[14] S.Heikkila and V. Lakshmikantham, Monotone Iterative Technique for Nonlinear Discontinues Differential Equations, Marcel Dekker Inc., NewYork, 1994.

[15] X. Hu, J Yan, The global attractivity and asymplotic stability of solution of a nonlinear integral equation, J. Math. Anal. Appl 321(2006), 147-156.