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Solution of Space Fractional Radon Diffusion Equation in Soil Medium

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Abstract: In this paper we present the Crank-Nicolson finite difference scheme for space fractional radon diffusion equation (SFRDE) in soil medium. We discuss that the scheme is unconditionally stable and convergence of the scheme is also verified at the length. Validation of the solution is carried out with the help of graphical illustration using 'Mathematica' software.

***Index Terms* -** Fractional calculus, Grunwald formula, Stability, Convergence, and Mathematica.

I. INTRODUCTION

The Fractional Calculus (FC) is a generalization of classical calculus concerned with operations of integration and differentiation of non-integer (fractional) order. The concept of fractional operators has been introduced almost simultaneously with the development of the classical ones. The study of fractional calculus has been a highly specialized and isolated field of mathematics. The fractional calculus was recognized to represent an useful tool for understanding and modeling many natural and artificial phenomena. Fractional calculus has many applications in biology, physics, engineering, economics etc. [1,2,6]. Most of the fractional differential equations do not have analytical solution therefore approximation and numerical techniques are developed. There are many numerical methods to find the solution of classical differential equations, while numerical methods for the fractional differential equations are very limited. As the fractional derivatives are the generalization of classical derivatives, the numerical techniques for the classical differential equations can be extended to the fractional differential equations in some way. In the recent years, there are many numerical techniques like finite difference method (FDM), finite element method (FEM), He's variational iterational method, Adomian decomposition method (ADM), matrix transform method (MTM), etc. Finite difference method is very rich and continuous to be developed. Also this method is very powerful tool and widely used to solve the differential equations as well as fractional differential equations in science and engineering. The main cause of implementation of this method is simple and easy to be put into practice in computer programs. Many papers have recently published on finite difference methods for solving the diffusion equation [3,4,78,9,10,11].

In this paper we discuss the fractional radon diffusion equation in soil medium. The diffusion theory came from the famous physiologist Adolf Fick. He stated that the flux density J is proportional to the gradient of concentration. This gives,

$$\mathrm{J} = -D\frac{\partial \mathrm{C}}{\partial \mathrm{t}} \tag{1.1}$$

where J is the radon flux density is diffusion coefficient , $\frac{\partial \mathrm{C}}{\partial \mathrm{t}}$ is gradient of radon concentration and D is diffusivity coefficient of radon. Now the change in concentration to change in time and position is stated by the Fick's second law which is the extension of Fick's first law, that gives,

$$\frac{\partial \mathrm{C(x,t)}}{\partial \mathrm{t}} = \frac{\partial^2 \mathrm{C(x,t)}}{\partial t^2} - \lambda \mathrm{c(x,t)} \tag{1.2}$$

where λ = $2.1{\times}10^{-6}s^{-1}$ is the decay constant. Many researchers have discussed the radon transport through soil, activated charcoal, concrete, etc. [5,12,13,14,15,16].

Here, we develop the space fractional crank-nicolson finite difference method for fractional order RDE in soil medium. We consider the following space fractional radon diffusion equation [SFRDE],

$$\frac{\partial C(x,t)}{\partial t} = D\frac{\partial^\beta C(x,t)}{\partial t^\beta} - \lambda C(x,t), 0 < x < L,\ 1 < \beta \le 2, t \ge 0\,, (x,t) \in [0,L] \times [0,T] \tag{1.3}$$

initial conditions: $C(x,0) = 0, 0 < x < L$ (1.4)

boundary conditions: $C(0,t) = c_0$ and $\frac{\partial C(x,t)}{\partial t} = 0, t \ge 0$ (1.5)

Definition1.1:-The Grunwald Letnikov space fractional derivative of order β is defined by,

$$\frac{\partial^\beta C(x,t)}{\partial x^\beta} = oD_x^\beta C(x,t) = \frac{1}{h^\beta}\lim_{N\to\infty}\sum_{j=0}^{N}\frac{\Gamma(j-\beta)}{\Gamma(-\beta)\Gamma(j+1)}C(x-(j-1)h,t)$$
$$= \frac{1}{h^\beta}\lim_{N\to\infty}\sum_{j=0}^{N} g_{\beta,j}C(x-(j-1)h,t)$$

where

$$g_{\beta,j} = \frac{\Gamma(j-\beta)}{\Gamma(-\beta)\Gamma(j+1)}$$

We organize the paper as follows: Section 2 is devoted for to develop Crank-Nicolson finite difference scheme for space fractional radon diffusion equation. In section 3, we discuss the stability of the approximated solution obtained by Crank-Nicolson finite difference scheme developed for fractional radon diffusion equation. In section 4, we discuss the convergence of the scheme. In the last section we solved test problem and their solution is represented graphically by mathematical software Mathematica.

II. FINITE DIFFERENCE SCHEME

In this section, we develop the space fractional Crank-Nicolson finite difference method for fractional order radon diffusion equation (1.3)-(1.5).

We define,

$$t_k = k\tau\,;\ k = 0,1,2,\dots,N \ and\ x_i = ih\,;\ i = 0,1,2,\dots,M$$

where

$$\tau = \frac{T}{N} \ and\ h = \frac{L}{M}$$

Let $C(x_{i,}t_k); i = 0,1,2,\dots,M$ and $k = 0,1,2,\dots,N$ be the exact solution of space fractional radon diffusion equation (SFRDE) (1.3)-(1.5) at mesh point$(x_{i,}t_k)$. Let c_i^k be the numerical approximation of the point $C(x_{i,}t_k)$.

We consider the spatial $\beta - order$ fractional derivative using the Grunwald finite difference formula at all-time levels. The standard Grunwald estimates generally yields unstable finite difference equation regardless of whatever results in finite difference method is an explicit or implicit system for related discussion. Therefore, we use a right shifted Grunwald formula to estimate the spatial $\beta - order$ fractional derivative.

For

$$\frac{\partial^\beta C(x,t)}{\partial x^\beta} = oD_x^\beta C(x,t) = \frac{1}{h^\beta}\sum_{j=0}^{i+1} g_{\beta,j}C\big(x_{i-(j-1)h}, t_{k+1}\big) + O(h^2)$$

and the normalized Grunwald weights are given by,

$$g_{\beta,0} = 1\ and\ g_{\beta,j} = (-1)^j\frac{\beta(\beta-1)\dots(\beta-j+1)}{j!}\ , j = 1,2,3,\dots$$

Using forward difference formula for time, right shifted Grunwald formula for second order space. Therefore, Crank-Nicolson type numerical approximation to equation (1.3) is given as follows-

$$\left[\frac{C_i^{k+1} - C_i^k}{\tau}\right] = D\frac{1}{2h^\beta}\left[\sum_{j=0}^{i+1} g_{\beta,j}\, C_{i-j+1}^{k+1} + \sum_{j=0}^{i+1} g_{\beta,j}\, C_{i-j+1}^{k}\right] - \lambda\, C_i^{k+1}$$

$$\left[C_i^{k+1} - C_i^k\right] = D\frac{\tau}{2h^\beta}\left[\sum_{j=0}^{i+1} g_{\beta,j}\, C_{i-j+1}^{k+1} + \sum_{j=0}^{i+1} g_{\beta,j}\, C_{i-j+1}^{k}\right] - \lambda\,\tau\, C_i^{k+1}$$

let

$$r = D\frac{\tau}{2h^\beta}\ and\ \mu = \lambda\,\tau$$

$$[C_i^{k+1} - C_i^k] = r\left[\sum_{j=0}^{i+1} g_{\beta,j}\, C_{i-j+1}^{k+1} + \sum_{j=0}^{i+1} g_{\beta,j}\, C_{i-j+1}^{k}\right] - \mu\, C_i^{k+1}$$

$$(1+\mu)C_i^{k+1} - r\,\sum_{j=0}^{i+1} g_{\beta,j}\, C_{i-j+1}^{k+1} = C_i^k + r\sum_{j=0}^{i+1} g_{\beta,j}\, C_{i-j+1}^{k} \tag{2.1}$$

$$(1+\mu)C_i^{k+1} - r g_{\beta,1} C_i^{k+1} - r\sum_{j=0, j\neq 1}^{i+1} g_{\beta,j}\, C_{i-j+1}^{k+1} = C_i^k + r g_{\beta,1} C_i^k + r\sum_{j=0, j\neq 1}^{i+1} g_{\beta,j}\, C_{i-j+1}^{k}$$

since $g_{\beta,1} = -\beta$

$$(1+\mu+r\beta)C_i^{k+1} - r\,\sum_{j=0, j\neq 1}^{i+1} g_{\beta,j}\, C_{i-j+1}^{k+1} = (1-r\beta)C_i^k + r\sum_{j=0, j\neq 1}^{i+1} g_{\beta,j}\, C_{i-j+1}^{k} \tag{2.2}$$

Therefore, the complete discretized problem is:

$$(1+r\beta+\mu)\, C_i^1 - r\sum_{j=0,j\neq 1}^{i+1} g_{\beta,j}\, C_{i-j+1}^{1} = (1-r\beta)C_i^0 + r\sum_{j=0,j\neq 1}^{i+1} g_{\beta,j}\, C_{i-j+1}^{0}, \qquad for\ k=0 \tag{2.3}$$

$$(1+r\beta+\mu)C_i^{k+1} - r\sum_{j=0,j\neq 1}^{i+1} g_{\beta,j}\, C_{i-j+1}^{k+1} = (1-r\beta)C_i^{k} + r\sum_{j=0,j\neq 1}^{i+1} g_{\beta,j}\, C_{i-j+1}^{k} \qquad for\ k\geq 1 \tag{2.4}$$

initial conditions, $C_i^0, i = 0,1,2,\dots,M$ (2.5)

boundary conditions, $C_0^k = C_0$ and $C_{M+1}^k = C_{M-1}^k.; k = 0,1,2,\dots,N$ (2.6)

and $r = D\frac{\tau}{2h^\beta}$; $\mu = \lambda\,\tau$

The matrix form of the above initial boundary value problem is

$$\mathrm{AC}^1 = \mathrm{BC}^0 + \mathrm{S}\,;\ \mathrm{for\ k} = 0 \tag{2.7}$$

$$\mathrm{AC}^{\mathrm{k}+1} = \mathrm{BC}^{\mathrm{k}} + S',\ for\ k\ \geq 1 \tag{2.8}$$

where

$$A = \begin{pmatrix} 1+r\beta+\mu & -rg_{\beta,0} & \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots \\ -rg_{\beta,2} & 1+r\beta+\mu & -rg_{\beta,0} & \cdots\cdots\cdots & \cdots\cdots\cdots & \vdots \\ -rg_{\beta,3} & -rg_{\beta,2} & 1+r\beta+\mu & -rg_{\beta,0} & \cdots\cdots\cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ -rg_{\beta,M} & -rg_{\beta,M-1} & \cdots\cdots & \cdots\cdots\cdots & -r(g_{\beta,0}+g_{\beta,2}) & 1+r\beta+\mu \end{pmatrix};$$

$$B = \begin{pmatrix} (1-r\beta) & rg_{\beta,0} & \cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots \\ rg_{\beta,2} & (1-r\beta) & rg_{\beta,0} & \cdots\cdots\cdots & \cdots\cdots\cdots & \vdots \\ rg_{\beta,3} & rg_{\beta,2} & (1-r\beta) & -rg_{\beta,0} & \cdots\cdots\cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ rg_{\beta,M} & rg_{\beta,M-1} & \cdots\cdots & \cdots\ \cdots\cdots & r(g_{\beta,0}+g_{\beta,2}) & (1-r\beta) \end{pmatrix};$$

$$S = \begin{pmatrix} rg_{\beta,2}C_0^1 + rg_{\beta,2}C_0^0 \\ rg_{\beta,3}C_0^1 + rg_{\beta,3}C_0^0 \\ rg_{\beta,4}C_0^1 + rg_{\beta,4}C_0^0 \\ \vdots \\ \vdots \\ rg_{\beta,M+1}C_0^1 + rg_{\beta,M+1}C_0^0 \end{pmatrix}; S' = \begin{pmatrix} rg_{\beta,2}C_0^{k+1} + rg_{\beta,2}C_0^k \\ rg_{\beta,3}C_0^{k+1} + rg_{\beta,3}C_0^k \\ rg_{\beta,4}C_0^{k+1} + rg_{\beta,4}C_0^k \\ \vdots \\ \vdots \\ rg_{\beta,M+1}C_0^{k+1} + rg_{\beta,M+1}C_0^k \end{pmatrix}$$

$$C^k = [c_1^k, c_2^k, c_3^k, \dots\dots\dots., c_N^k]^T; r = D\frac{\tau}{2h^\beta}\ ;\ \mu = \lambda\,\tau;, i = 0,1,2,\dots,M; k = 0,1,2,\dots,N;$$

$$g_{\beta,0} = 1\ and\ g_{\beta,j} = (-1)^j\,\frac{\beta(\beta-1)\dots(\beta-j+1)}{j!}\ , j = 1,2,3,\dots$$

III. STABILITY

Theorem 3.1: The solution of approximated initial boundary value problem (2.3)-(2.6) for space fractional radon diffusion equation (SFRDE) (1.3)-(1.5) is unconditionally stable.

Proof: We assume that, $||E^k||_\infty \leq |\epsilon_l^k|\ = \max_{1\leq i\leq M} \epsilon_i^k$

Therefore, for k=0, from equation (2.3), we get

$$|\epsilon_l^1| = \left|(1 + r\beta + \mu)\epsilon_i^1 - r\sum_{j=0, j\neq 1}^{i+1} g_{\beta,j}\, \epsilon_{i-j+1}^1\right|$$
$$=\left|(1 - r\beta)\epsilon_i^0 + r\sum_{j=0,j\neq 1}^{i+1} g_{\beta,j}\, \epsilon_{i-j+1}^0\right|$$
$$\leq \left|1 - r\beta + r\sum_{j=0,j\neq 1}^{i+1} g_{\beta,j}\right|\epsilon_l^0\,; \qquad \because (\, g_{\beta,1} = -\beta)$$
$$\leq \left|\epsilon_l^0\right|; \qquad \because \ (\sum g_{\beta,j} < 0 \ \Rightarrow 1 + r\sum g_{\beta,j} < 1)$$

Therefore,

$$||E^1||_\infty \leq ||E^0||_\infty$$

Thus, the result is true for k = 0.
Suppose that, the result is true for k,

$$||E^k||_\infty \leq ||E^0||_\infty$$

To prove that the result is true for k+1, from equation (2.4), we have

$$\left|\epsilon_l^{k+1}\right| = \left|(1 + r\beta + \mu)\epsilon_i^{k+1} - r\sum_{j=0, j\neq 1}^{i+1} g_{\beta,j}\, \epsilon_{i-j+1}^{k+1}\right|$$
$$=\left|(1 - r\beta)\epsilon_i^k + r\sum_{j=0,j\neq 1}^{i+1} g_{\beta,j}\, \epsilon_{i-j+1}^k\right|$$
$$\leq \left|1 - r\beta + r\sum_{j=0,j\neq 1}^{i+1} g_{\beta,j}\right|\epsilon_l^0\,; \qquad \because \ (\, g_{\beta,1} = -\beta)$$
$$\leq \left|\epsilon_l^0\right|, \qquad \because (\sum g_{\beta,j} < 0 \ \Rightarrow 1 + r\sum g_{\beta,j} < 1\,)$$

Therefore,

$$||E^{k+1}||_\infty \leq ||E^0||_\infty$$

Thus, the result is true for k+1.
Hence by mathematical induction, the result is true for all k.

$$||E^{k+1}||_\infty \leq ||E^0||_\infty$$

Thus, the scheme is unconditionally stable.

IV. CONVERGENCE

In this section, we discuss the convergence of the approximate finite difference scheme (2.3) - (2.6). Let $C(x_i, t_k)$ be the exact solution of the SFRDE (1.3)-(1.5) and C_i^k be the exact solution of the discrete equation (2.3)-(2.6) at the mesh point$(x_i, t_k), where\ i = 0,1, \dots, M-1; k = 1,2, \dots, N$.
We define,$e_i^k = C(x_i, t_k) - C_i^k, where\ i = 0,1, \dots, M-1; k = 1,2, \dots, N$ and $E^k = (e_1^k, e_2^k, \dots, e_{M-1}^k)$

Theorem 4.1 The fractional order Crank-Nicolson finite difference scheme (2.3)-(2.6) for SFRDE (1.3)-(1.5) is convergent and the solution C_i^k of the discretize scheme (2.3)-(2.6) and the solution $C(x_i, t_k)$ of the equation (1.3)-(1.5) satisfy,

$$||C(x_i, t_k) - C_i^k|| \ \leq ||E||_\infty + O\ (\tau + h^{2-\beta}); i = 0,1, \dots, M-1; k = 0,1, \dots, N$$

Proof: Let us assume that,

$$|e_l^k| \ = \max_{1\leq i\leq M-1} |\ \epsilon_i^k\ | = \left||E|\right|_\infty; for\ l = 1,2, \dots$$

and

$$T_l^k = \max_{1\leq i\leq N} |\,T_i^k\,|\,; T_j^n = h^2[O(\tau) + O(h^{2-\beta})]$$

Therefore, from equation (4.1), we have

$$|e_l^1| = \left|(1 + r\beta + \mu)e_i^1 - r\sum_{j=0, j\neq 1}^{i+1} g_{\beta,j}\, e_{i-j+1}^1\right|$$
$$=\left|(1 - r\beta)e_i^0 + r\sum_{j=0,j\neq 1}^{i+1} g_{\beta,j}\, e_{i-j+1}^0\right|$$
$$\leq \left|1 - r\beta + r\sum_{j=0,j\neq 1}^{i+1} g_{\beta,j}\right|e_l^0\,; \ since\ g_{\beta,1} = -\beta$$
$$\leq \left|e_l^0\right|, because\ \sum g_{\beta,j} < 0 \ \Rightarrow 1 + r\sum g_{\beta,j} < 1$$

Therefore,

$$||E^1||_\infty \le ||E^0||_\infty + h^2[O(\tau) + O(h^{2-\beta})]$$

Suppose that

$$||E^k||_\infty \le ||E^0||_\infty + h^2[O(\tau) + O(h^{2-\beta})]$$

From equation (2.4), we have

$$|e_l^{k+1}| = \left|(1 + r\beta + \mu)e_i^{k+1} - r\sum_{j=0,j\neq1}^{i+1} g_{\beta,j}\, e_{i-j+1}^{k+1}\right|$$
$$=\left|(1 - r\beta)e_i^k + r\sum_{j=0,j\neq1}^{i+1} g_{\beta,j}\, e_{i-j+1}^k\right|$$
$$\le \left|1 - r\beta + r\sum_{j=0,j\neq1}^{i+1} g_{\beta,j}\right| e_l^k \text{ ; } since\ g_{\beta,1} = -\beta$$
$$\le |e_l^k|, because\ \sum g_{\beta,j} < 0 \implies 1 + r\sum g_{\beta,j} < 1$$

Therefore,

$$||E^{k+1}||_\infty \le ||E^0||_\infty + h^2[O(\tau) + O(h^{2-\beta})]$$

Hence by mathematical induction, the result is true for all k.

$$||E^k||_\infty \le ||E^0||_\infty + h^2[O(\tau) + O(h^{2-\beta})]$$

This shows that fractional finite difference scheme (2.3)-(2.6) for SFRDE (1.3)-(1.5) is convergent.

V. NUMERICAL SOLUTION

The approximated solution of space fractional radon diffusion equation in soil medium with initial and boundary conditions is achieved. The numerical solution of the space fractional radon diffusion equation (SFRDE) by the finite difference scheme is validated by using software, it is important to use some analytical model. Therefore, we have solved the problem at specific particular conditions by using Mathematica Software. We consider the following, dimensionless space fractional radon diffusion equation with suitable initial and boundary conditions.

$$\frac{\partial C(x,t)}{\partial t} = D\frac{\partial^\beta C(x,t)}{\partial t^\beta} - \lambda C(x,t), 0 < x < L, 1 < \beta \le 2, t \ge 0\,, (x,t) \in [0,L] \times [0,T]$$

initial condition: $C(x,0) = 0{,}0 < x < L$

boundary conditions: $C(0,t) = c_0\ and\ \frac{\partial C(x,t)}{\partial t} = 0, t \ge 0$

with the radon diffusion coefficient $D = 4.1 \times 10^{-7}\ m^2/s$. The numerical solutions obtained at t = 0.05 by considering the parameters $L = 1.7278cm$, $\lambda = 2.1 \times 10^{-6}s^{-1}, \tau = 0.05, k = 4\,m^2/kg,$ $\rho = 0.5\ g/cm^3$, $c_0 = 200\ Bq/m^3$, $c(0,t) = 40 \times 10^3$, $\alpha = 0.9, \alpha = 0.8$ is simulated in the following figure,

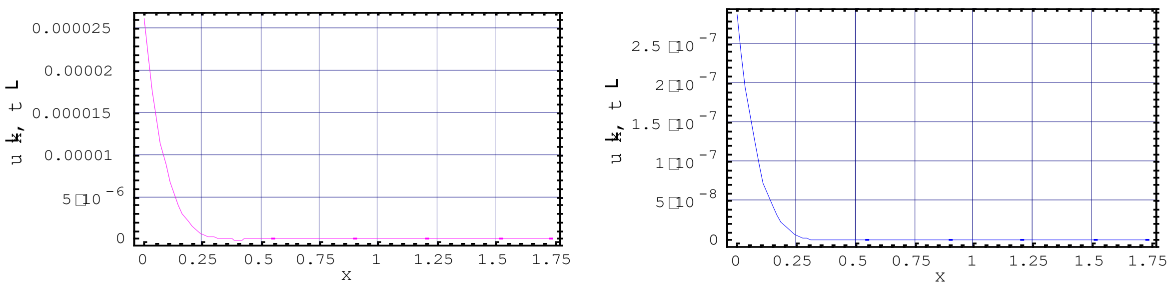

Fig. The approximate solution of radon diffusion equation for $\alpha = 0.9\ and\ \alpha = 0.8$

VI. CONCLUSION

We successfully develop the fractional order Crank-Nicolson finite difference scheme for space fractional radon diffusion equation. Furthermore we discuss its stability and convergence of the scheme. As an application of this method we obtain the numerical solution of text problem and its solution is simulated graphically by mathematical software Mathematica.

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