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A Vizing Type Theorem on Maximum Degree and Minimum Degree

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Abstract : Vizing in 1965 providede an upper bound for the number of edges in terms of domination number and is popular as Vizing Theorem. In this paper we find a bound on the number of edges in a simple graph when the maximum degree Δ (minimum degree δ) is given. This result corresponds to the well known theorem of Vizing. We have characterized the graphs for which these bounds are attained. We also give a lower bound for δ in terms of order and size of the graph.

IndexTerms – ***Maximum degree Δ, Minimum degree δ, Partially complete (p, δ) graph, k-semiregular graph.***

I. Introduction

For any undefined terminologies we refer (Harrary 1969). In our discussion by $G(p, q)$ we mean a graph with p vertices and q edges. The set of p vertices is denoted by V and set of edges is denoted by E. For any vertex $v \in V$, $N(v) = \{u \in V \mid u$ is adjacent to $v\}$. The degree of v denoted as $d(v) = |N(v)|$. We here quote some well known results for our reference. The first and foremost result is popularly known as First Theorem in Graph Theory which some times called as Hand shaking Lemma

Theorem 1.1. (Harrary 1969). *For any graph G(p,q) we have* $\sum_{v \in V} d(v) = 2q$

An immediate consequence of the above theorem is the following

Corollary 1.1.1. (Harrary 1969). *In any graph the number of odd degree vertices is even.*

Let $\Delta(G)$ and $\delta(G)$ respectively denote the maximum degree and minimum degree of G. It is well known that the average degree of a vertex v lies between minimum and maximum degree of v and hence

Theorem 1.2 (Harrary 1969). For any graph *G(p,q) with maximum degree* $\Delta(G)$ *and minimum degree* $\delta(G)$,

$$\delta(G) \leq \frac{2q}{p} \leq \Delta(G) \tag{1}$$

The domination number of a graph is well studied concept in graph theory. For a detailed study on domination one can refer (Haynes et al. 1999), (Hedetnimi et al. 2006), (Fink et al. 1985). We say that two vertices dominate each othe if they are adjacent. The *domination number* $\gamma(G)$ is the minimum number of vertices needed to dominate all the vertices of G. (Vizing 1965) gave the following bound on the number of edges when the domination number $\gamma(G)$ is given.

Theorem 1.3 [3]. *Let G be any graph of order* p , *size* q *and* $\gamma(G) \geq 2$, then

$$q \leq \frac{(p-\gamma(G))(p-\gamma(G)+2)}{2} \tag{2}$$

On the similar lines (Rautenbach 1968) extended this result for weak domination number $\gamma_w(G)$.

Theorem 1.4 [2]. *Let G be any graph of order* p , *size* q *and* $\gamma_w(G) \geq 2$, then

$$q \leq \frac{p(p-1)-\gamma_w(\gamma_w-1)}{2} \tag{3}$$

In this paper we find a bound on the number of edges in a simple graph when the maximum degree Δ or minimum degree δ is given. This result corresponds to the theorem of Vizing. We have characterized the graphs for which these bounds are attained by defining two types of graphs viz, *k*-semiregular graphs and partially complete (p, δ) graphs. We also give a lower bound for δ in terms of order and size of the graph.

To begin with we give a bound on number of edges when the maximum degree Δ or *minimum degree* δ is given.

Proposition 1.5 *Let G be any graph of order p , size q, maximum degree Δ and minimum degree δ. Then*

$$\Delta \leq q \leq \left\lfloor \frac{p\Delta}{2} \right\rfloor \qquad (4)$$

$$\left\lceil \frac{p\delta}{2} \right\rceil \leq q \leq \frac{p(p-3)+2(1+\delta)}{2} \qquad (5)$$

Proof. The upper bound in (4) follows from the fact that $\frac{2q}{p} \leq \Delta(G)$. If v is a vertex with maximum degree $\Delta(G)$ then there exists atleast Δ edges incident on v and hence the total number of edges $q \geq \Delta$ and hence lower bound follows. The lower bound in (5) is immediate from the fact that $\delta(G) \leq \frac{2q}{p}$. Let v be a vertex of minimum degree δ. Then v is adjacent to δ vertices and these δ vertices can be at most of degree p - 1. The remaining $p - \delta - 1$ vertices can be mutually adjacent to each other except the vertex v. Thus these $p - \delta - 1$ vertices are at most of degree p - 2.Then $2q \leq \delta(p-1) + (p - \delta - 1)(p-2) + \delta = p(p-3) + 2(\delta+1)$. Hence the upper bound in (2) follows. Any regular graph attains the lower bound in (2). The complete graph *Kp* attains the upper bound in (5). ■

From the above proposition we get a lower bound for minimum degree in terms of order and size of the graph.

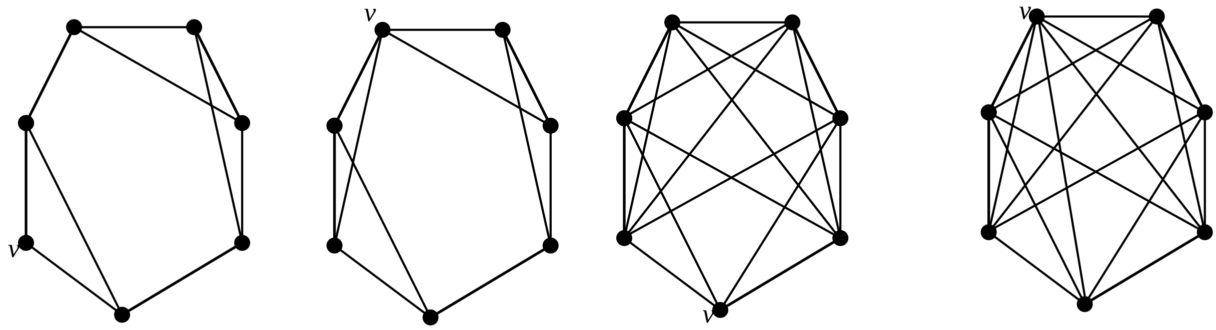
Corollary 1.5.1. *Let G be any graph of order and , minimum degree δ.*

Then
$$\frac{2q - p(p-3) - 2}{2} \leq \delta \qquad (6)$$

Proof. From Proposition 1.5, we have $q \leq \frac{p(p-3)+2(1+\delta)}{2}$. This yields $2q - p(p-3) - 2 \leq 2\delta$ which yields the desired bound. Further any complete graph *Kp* attains the bound.

II. K-SEMIREGULAR GRAPHS

Definition 2.1. A graph G is said to be *k-semi regular* if all the vertices are of degree k except one vertex of degree $k \pm 1$. We thus have two class of *k-semi regular* graphs. A graph G is said to be *k-semi regular graph of first kind* if all the vertices in G are of degree k except one vertex of degree k - 1. On the other hand a graph G is said to be *k-semi regular graph of second kind* if all the vertices are of degree k except one vertex of degree $k + 1$. The *k*-semiregular graphs of both kinds for $k = 3, 5$ is shown in the Fig.1.1. The vertex v shown in each figure represents the one vertex of degree $k \pm 1$. Immediately we observe that the number of edges in any *k*-semiregular graph of first kind is $\frac{pk-1}{2}$ and that of second kind is $\frac{pk+1}{2}$. Hence if G is any *k*-semiregular graph of first kind by joining the minimum degree vertex v with any other nonadjacent vertex we get a *k*-semiregular graph of second kind.



3-Semiregular graph of first kind | 3-semiregular graph of second kind | 5-Semiregular graph of first kind | 5-Semiregular graph of second kind

Fig.1.1

Proposition 2.1 *Every k-semi regular graph is of odd order.*

Proof. Let G be a k-semiregular graph with p vertices. Suppose p is an even number. Then pk is even for any k and hence $pk \pm 1$ is odd. Since G is k semiregular, there exist p-1 vertices of degree k and one vertex of degree $k \pm 1$.
Hence if d(v) is the degree of a vertex then by the well known first theorem in Graph theory we have $2q = \sum_{v \epsilon V} d(v) = (p\text{-}1)k + k \pm 1 = pk \pm 1$ which is an odd number – a contradiction. Hence G must be of odd order.

Proposition 2.2 *If G is a k-semiregular graph then k is odd.*

Proof. Let G be a k-semiregular graph with p vertices. Then from the Proposition 1, p is an odd number. Suppose k is even. Then $k \pm 1$ is odd. Since G is k-semiregular, there exist p-1 vertices of degree k and one vertex of odd degree $k \pm 1$. This is a contradiction to the fact that the number of odd degree vertices in any graph is even. Hence we conclude that k must be an odd number.

We now characterize the graphs for which the bound is attained in (4).

Proposition 2.3 *Let G be any graph of order p, size q, and maximum degree* Δ.

Then (i). $q = \left\lfloor \frac{p\Delta}{2} \right\rfloor$ *if and only if G is regular or k - Semi regular graph of first kind.*

(ii). $q = \Delta$ *if and only if* $G = K_{1,n} \cup \overline{K}_{p-n-1}$.

Proof. (i). If G is regular then we have $2q = p\Delta$. If G is k - Semi regular graph of first kind then there exist p-1 vertices of degree $\Delta = k$ and one vertex of degree Δ-1. Hence $2q = (p\text{-}1)\Delta + \Delta - 1 = p\Delta - 1$. Thus in any case $q = \left\lfloor \frac{p\Delta}{2} \right\rfloor$ holds. Conversely, Suppose $q = \left\lfloor \frac{p\Delta}{2} \right\rfloor$. Then 2q = $p\Delta$ or $p\Delta$ -1 . If 2q = $p\Delta$ then G is Δ- regular. In the latter case we have $2q = p\Delta - 1 = p\Delta - \Delta + \Delta - 1 = (p - 1)\Delta + \Delta - 1$. This implies that there are (p-1) vertices of degree $\Delta = k$ and one vertex of degree Δ -1. Hence G is k- semi regular graph (ii). If $G = K_{1,n} \cup \overline{K}_{p-n-1}$ then it is not hard to see that $q = \Delta$. Conversely if $q = \Delta$ then there exist at most one vertex of degree Δ and Δ vertices of degree 1 and all the remaining vertices of degree 0. This implies that G is $K_{1,n} \cup \overline{K}_{p-n-1}$. ■

The 3-Semi regular graph of first kind with q =10, and 5-semiregular graph of first kind with q =17 on p = 7 vertices shown in the Fig.1 satisfy the bound $q = \left\lfloor \frac{p\Delta}{2} \right\rfloor$.

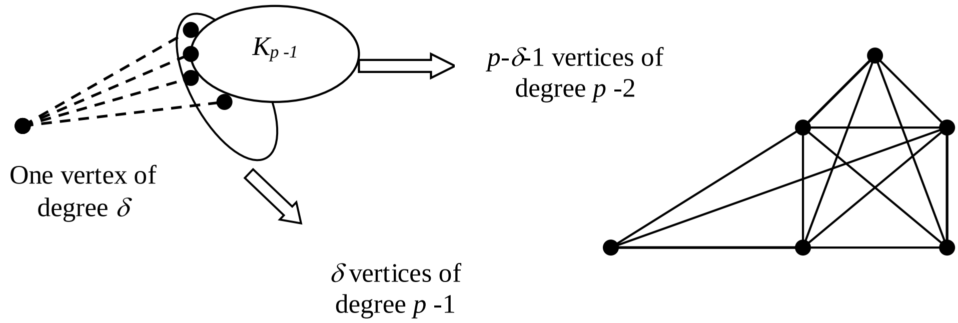
III. PARTIALLY COMPLETE (*p*, δ) GRAPH

Definition 3.1. A graph G is said to be a *partially complete* (p, δ) graph if G is obtained by identifying a vertex v in K_p and removing k edges ($0 \le k \ge p$-1) incident on v. Any partially complete graph on p vertices is denoted by K_{p-1}^{δ}. When $k = 0$ we get $\delta = p$ -1 and hence G is a complete graph. Thus K_p is a partially complete graph K_{p-1}^{p-1}.

Any partially complete graph posses the following properties.

1. K_{p-1}^{δ} has K_{p-1} as induced subgraph.
2. K_{p-1}^{δ} has exactly one vertex of degree $\delta = p\text{-}k\text{-}1$ and $p\text{-}\delta\text{-}1$ vertices of degree p-2, and δ vertices of degree p-1.
3. The number of edges in K_{p-1}^{δ} is equal to $q = \frac{p(p-3) + 2(1+\delta)}{2}$.
4. Given any $p \ge 2$ there exist p distinct complete (p, δ) graphs $K_{p-1}^{0}, K_{p-1}^{1}, K_{p-1}^{2}, \ldots\ldots, K_{p-1}^{p-1}$ of size ${}^{p-1}C_2$, ${}^{p-1}C_2 + 1$, ${}^{p-1}C_2 + 2$, ... ${}^{p}C_2$ respectively.

The schematic representation of partially complete (p, δ) graph and the graph K_5^3 is shown in the Fig.2.



K_{p-1}^{δ} Fig .2 K_5^3

A schematic representation of partially complete (p,δ) graph K_{p-1}^{δ}

We next characterize the graphs for which the bounds are attained in (5).

Proposition 3.1. *Let G be any graph of order p, size q and minimum degree* δ. *Then*

(i) $q = \left\lceil \frac{p\delta}{2} \right\rceil$ *if and only if G is a regular or k - semi regular graph of second kind.*

(ii) $q = \frac{p(p-3)+2(1+\delta)}{2}$ *if and only if* $G \cong K_{p-1}^{\delta}$.

Proof. To prove (i). If G is regular then $2q = p\delta$. If G is k - semi regular graph of second kind then there exist p-1 vertices of degree $\delta = k$ and one vertex of degree k+1. Then $2q = (p-1)\,\delta + \delta + 1 = p\delta + 1$. Hence in any case $q = \left\lceil \frac{p\delta}{2} \right\rceil$ holds. Conversely, Suppose $q = \left\lceil \frac{p\delta}{2} \right\rceil$ holds. Then $2q = p\delta$ or $p\delta + 1$. If $2q = p\delta$ we have G is a graph in which all the vertices are of minimum degree and hence G is a regular graph. If $2q = p\delta + 1 = p\delta - \delta + \delta + 1 = (p-1)\,\delta + (1+\delta)$, this implies G has p-1 vertices of degree $\delta = k$ and one vertex of degree $\delta + 1$. Then G is a k-semiregular graph of second kind.

To prove (ii). If $G \cong K_{p-1}^{\delta}$, then from the property of a partially complete graph, G has $q = \frac{p(p-3)+2(1+\delta)}{2}$. Conversely, let the number of edges in G be $q = \frac{p(p-3)+2(1+\delta)}{2}$. We first observe that to have maximum number of edges we must have only one minimum degree vertex. Let v be a vertex of minimum degree δ. Then as in the proof of Proposition 4, there exist δ vertices exactly of degree p - 1. The remaining $p - \delta - 1$ vertices are exactly of degree p - 2. Then G must be K_{p-1}^{δ}.

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