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# Convolution of Generalized Functions

**P N Pathak***

Department Of Mathematics, SBS, CSJM University, Kanpur

Email id: drpnpathak@csjmu.ac.in

**Abstract**: In this paper we first explore the theory of convolution of generalized functions (Distributions) with some important properties two specific cases of convolution of Generalized functions namely Legendre transformable functions and convolution of Generalized Laguerre transformable functions has been discussed. Finally the established Convolution theorem by Churchill and Dolph has been extended for generalized Legendre transformable functions.

**Introduction**: One of the important problem in the theory of integral transformation is that, knowing the transforms of two given functions F(x) and G(x) namely *f*(n) and g(n) respectively, to know the inverse transform of the product *f*(n).g(n). This also means that finding a function H(x) in terms of F(x) and G(x) who transform is *f*(n).g(n). The result concerning such problem is known as *convolution*.

The convolution of distributions extends the classical operation to a broader class of mathematical objects such as testing functions and Generalized functions (distributions). In this consequences we first define the convolution of a distribution and a test function and then the convolution of two distributions:

The convolution of a distribution $T\in D'(Rn)$ with a compactly supported smooth function

$\phi\in C^\infty(Rn)$ is defined as

$$\langle T*\phi\rangle\,(x)=\langle Ty,\phi(x-y)\rangle. \qquad (1)$$

In this expression, Ty indicates that the distribution acts on the test function with respect to the variable y, while x is treated as a parameter. A crucial property of this operation is that it "regularizes" the distribution. That is, even if T is a highly singular distribution, the resulting function $(T*\phi)(x)$ is infinitely differentiable ($C^\infty$). This smoothing effect is a direct consequence of the continuous linear action of the distribution on the smooth test function.

In general, the convolution of two distributions, S and T, is not always defined. It exist under the condition that the pairing $\langle Sx\otimes Ty,\ \phi(x+y)\rangle$ must be well-defined and continuous for every test function $\phi$. Therefore the common sufficient condition for existence is that one of the distributions must have compact support. The formal definition is given by the expression-

$$\langle S*T,\phi\rangle=\langle Sx\otimes Ty,\phi(x+y)\rangle. \qquad (2)$$

This formulation uses the tensor product of distributions, which is a powerful tool for defining operations on objects in multiple dimensions.

**Basic Properties:**

The extension of convolution to distributions raises important questions about whether the familiar algebraic properties from classical analysis still hold.

**Commutativity:** Convolution of distributional is commutative i.e.f∗g=g∗f.

**Associativity:** Convolution of distributional is associative under certain conditions i.e ⟨f∗g⟩∗h=f∗⟨g∗h⟩ holds for distributions when all three have compact support or satisfy other specific support conditions.

Generally the associative property dose not hold for all tempered distributions. The failure of associativity reveals a deeper and more complex algebraic structure in the space of distributions compared to classical function spaces. It serves as a critical reminder that extending operations from one mathematical domain to another is not always straightforward and requires careful verification of the conditions under which properties hold.

**Existence of Identity Element:** It is a unique and important property of distributional convolution is the role of the Dirac delta function called the identity element. The convolution of any distribution T with the Dirac delta function leaves the distribution unchanged: T∗δ=T. This property is a direct consequence of the delta function's "sifting" property and is a cornerstone of LTI system theory, where it represents a system that passes an input signal without change. The elegance of this property solidifies the delta function's place as a fundamental component of the theory of distributions.

**Classical vs. Distributional Convolution**

| **Property** | **Classical Functions** | **Distributions** |
|---|---|---|
| **Commutativity** | f∗g=g∗f always holds if the integral exists. | S∗T=T∗S holds when the convolution is defined. |
| **Associativity** | ⟨f∗g⟩∗h=f∗⟨g∗h⟩ always holds. | ⟨S∗T⟩∗U=S∗⟨T∗U⟩ holds under specific support conditions (e.g., both with compact support), but not in general. |
| **Existence** | Requires integrability or other conditions on the functions. | Not always defined for any pair; requires specific support conditions (e.g., one distribution having compact support). |
| **Regularizing Effect** | No inherent smoothing; a convolution of discontinuous functions can be continuous. | Convolving a distribution with a smooth test function produces an infinitely differentiable ($C^\infty$) function. |
| **Identity Element** | The Dirac delta function, δ, is the identity element, but it is not a classical function. | The Dirac delta function, δ, is a well-defined distribution that serves as the identity element. |

Following Zemanian [1], Pathak [9,11] we deal with the theory of convolution of generalized functions with some important properties two specific cases of convolution of Generalized functions namely Legendre transformable functions and convolution of Generalized Laguerre transformable functions has been discussed. Finally Convolution theorem established by Churchill and Dolph [2] has been extended for generalized Legendre transformable functions.

**Convolution of Laguerre Transformable Generalized Functions**

The convolution product $f{*}g$ of two generalized functions $f$ and g in $A'(L_0)$ is defined by the relation -

$$<f{*}g, \phi> = < f(x), < g(y), \phi(x+y) > > \qquad (3)$$

for every $\phi \in A(L_0)$. If we write $\psi(x) = < g(y), \phi(x+y) >$

Then following Zemanian [4], it can be easily proved that $\psi \in A'(L_0)$ when $\phi \in A(L_0)$. Finally it can be shown that $f * g$ as defined by (3) is also a member of $A'(L_0)$.

We now defined generalized Laguerre transform T of the convolution of $f$ and g in $A'(L_0)$ as follows.

$$T\{f\} = F(n) = (f, L_n(x)e^{-x/2}) \tag{4}$$

From (4) we have

$$T\{f*g\} = (f*g, L_n(t).e^{-t/2})$$

$$= (f(x) \otimes g(y), L_n(x+y)e^{-(x+y)/2}) \tag{5}$$

In the view of the following results

$$L_n(t) = L_n'(t) - L_{n-1}'(t)$$

and $$L_n'(x+y) = \sum_{k=0}^{n} L_{n-k}(x) L_k(y)$$

The relation (5) becomes

$$T\{f*g\} = \sum_{k=0}^{n} \left(f(x), L_{n-k}(x)\, e^{-x/2}\right)\left(g(y), L_k(y)\, e^{-y/2}\right) - \sum_{k=0}^{n-1} \left(f(x), L_{n-k-1}(x)\, e^{-x/2}\right)\left(g(y), L_k(y)\, e^{-y/2}\right) \tag{6}$$

Further if we assume that

$$f = \sum_{n=0}^{\infty} a_n\, L_n(t)\, e^{-t/2} \quad \text{and } g = \sum_{n=0}^{\infty} b_n\, L_n(t)\, e^{-t/2}$$

Where $a_n = (f, L_n(t)\, e^{-t/2})$ and $b_n = (g, L_n(t)\, e^{-t/2})$

Then the equation (6) reduces to

$$T\{f*g\} = \sum_{k=0}^{n} \left[b_k\, (a_{n-k} - a_{n-k-1})\right] \tag{7}$$

Churchill and Dolph [3] have obtained the convolution of two Legendre transformable functions by two different methods. In [3] they have obtained the convolution by finding the inverse transform of the product of transforms and in [2] by writing the product of transforms as transform of the function which turns out to the convolution.

By applying suitable transformations of variables they have written the convolution in different forms given below. The convolution H of F and G may be written as;

$$H(\cos\mu) = \frac{1}{\pi}\int_0^{\pi} F(\cos\lambda)\sin\lambda\, d\lambda \quad \times \int_0^{\pi} G(\cos\lambda\cos\mu + \sin\lambda\sin\cos\alpha)d\alpha, \tag{8}$$

or as

$$H(\cos\mu) = \frac{1}{\pi}\int_0^{\pi/2} \sin\phi\, d\phi \int_0^{2\pi} F[\sin\phi\sin(\beta - \frac{\mu}{2})] \times G[\sin\phi\sin(\beta + \frac{\mu}{2})]d\beta, \tag{9}$$

or as

$$H(x) = \frac{1}{\pi}\int\int_{E(x)} F(y)G(z)(1 - x^2 - y^2 - z^2 + 2xyz)^{-1/2}\, dydz \tag{10}$$

Where E(x) is the interior of the ellipse

$$y^2 + z^2 - 2xyz = 1 - x^2 \tag{11}$$

for each fixed x such that $-1 < x < 1$

The ellipse (11) may be written in the from

$$\frac{(z+y)^2}{2(1+x)} + \frac{(z-y)^2}{2(1-x)} = 1$$

the formula (10) can also be written in the form

$$H(x)=\frac{1}{\pi}\int_{-1}^{1} F(y)dy\int_{xy-\sqrt{(1-x^2)(1-y^2)}}^{xy+\sqrt{(1-x^2)(1-y^2)}} G(z)(1-x^2-y^2-z^2+2xyz)^{-1/2}\,dz \qquad (12)$$

In paper [3] they have derived the form (10) by finding the inverse transform of the product of transforms by using a sum of the series in product of three Legendre functions obtained by Vinti [5]. This process may be applied to obtain the convolution of Legendre Transformable generalized functions as shown below. Let

$a_n=(F(y),P_n(y))$,

$b_n=(G(y),P_n(z))$,

then we have

$$\sum_{n=0}^{\infty}(n+\frac{1}{2})a_n b_n P_n(x)$$

$$=\sum_{n=0}^{\infty}(n+\frac{1}{2})(F(y),P_n(y))(G(z),P_n(z))P_n(x)$$

$$=\sum_{n=0}^{\infty}(n+\frac{1}{2})(F(y)G(y),P_n(y)P_n(z))P_n(x)$$

$$=\sum_{n=0}^{\infty}(n+\frac{1}{2})(F(y)G(z),P_n(x)P_n(y)P_n(z))$$

$$=(F(y)G(z),\sum_{n=0}^{\infty}(n+\frac{1}{2})P_n(x)P_n(y)P_n(z))$$

$$=(F(y)G(z),\frac{1}{\pi}(1-x^2-y^2-z^2+2xyz)^{-1/2}), \qquad (13)$$

by the using of

$$P_n(x)P_n(y)=\frac{1}{\pi}\int_{xy-\sqrt{(1-x^2)(1-y^2)}}^{xy+\sqrt{(1-x^2)(1-y^2)}} P_n(z)(1-x^2-y^2-z^2+2xyz)^{(-1/2)}dz \qquad (14)$$

Hence the convolution may be obtained by the action of the direct product of generalized functions F and G to

$$\frac{1}{\pi}(1-x^2-y^2-z^2+2xyz)^{-1/2}.$$

In-fact (13) may be thought as the analytic continuation of (10). The above result is formally and we have to work further for precise justification of the process.

**Applications Convolution of Generalized Functions**

The convolution of generalized functions is not an abstract curiosity; it is a pragmatic and indispensable tool with wide-ranging applications across diverse scientific and engineering disciplines. The theory provides a rigorous foundation for a variety of formerly heuristic methods.

**In Partial Differential Equations**

The theory of distributions is crucial for solving linear partial differential equations. The method hinges on the concept of a **fundamental solution** (also known as a Green's function), which is a distribution that solves a differential operator with a Dirac delta function as its source term.

**In Signal and Image Processing**

The convolution of generalized functions is the foundation of modern digital signal processing. A cornerstone of this field is the principle that the output of any Linear Time-Invariant (LTI) system is the convolution of its input signal with its **impulse response**.

A modern and highly visible application is in **Convolutional Neural Networks (CNNs)**, which are a class of deep learning models used in computer vision and other fields. CNNs use a "convolutional layer" to extract features from an input, such as an image.

**In Probability Theory**

The convolution of distributions provides a deep and elegant explanation for a number of fundamental concepts in probability and statistics. One of the most important relationships is that the probability distribution of the sum of two independent random variables is the convolution of their individual probability distributions. This property provides a beautiful mathematical lens through which to understand the Central Limit Theorem.

**In Engineering and Science**

The output signal is the convolution of the input signal and the system's impulse response.

**In Computer Science**

A sliding window operation extracts features from an input, which is the core principle of convolutional layers in deep learning.

## Conclusion

The convolution of generalized functions represents a monumental advancement in mathematical analysis, providing a rigorous and powerful framework to unify classical concepts with singular, "ideal" objects that were previously treated heuristically. This elegant construction not only gives a precise meaning to concepts like the Dirac delta function but also enables operations like differentiation and convolution to be defined for a much wider class of objects.

The many algebraic properties of classical convolution, such as commutativity, have been extended to the distributional context, the theory also reveals important distinctions, most notably the failure of associativity for general tempered distributions.

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