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COUNTING AND BINOMIAL COEFFICIENTS IN GRAPH THEORY

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ABSTRACT

This paper summarizes aspects of language and mathematics that are not directly part of graph theory provide useful background for learning of graph theory in mathematics . this article explains counting and binomials coefficients in graph theory A discussion of counting quickly leads to summations and products. These can be written concisely using appropriate notation.We express summation using $\sum$, the uppercase Greek letter “sigma”. Vfhen a and b are integers, the value of $\sum_a^b f\ (i)$ is the sum of the numbers $f(i)$ over the integers i satisfying $a \leq i \leq b$. Here i is the index of summation, and the formula $f\ (i)$ is the summand .We write $\sum_{j\ \epsilon S} f\ (j)$ to sum a real-valued function f over the elements of a set S in its domain. When no subset is specified, as in $\sum_j x_j$, we sum over the entire domain. When the summand has only one symbol that can vary, we may omit the subscript on the summation symbol, as in $\sum x_j$. Similar comments apply to indexed products using $\prod$, which is the upper-case Greek letter pi$''$.Two simple rules help organizing the counting of finite sets by breaking problems into subproblems. These rules follow from the definition of size and properties of bijections.

Key words: summations , products , arrangement , selections , permutation , composition.

INTRODUCTION

Definition. The rule of sum states that if A is afinite set and B_1, B_m is a partition of A, then $|A| = \sum_{i=1}^{m} |\, B_i|$:

Let T be aset whose elements can be described using a procedure involving steps $S_1 \ldots\ldots S_k$ such that step S_j can be performed in r_j ways, regardless of *how* steps S_1, S_{j-1} are performed. The rule of product states that $\prod_{j=1}^{k} r_j$

For example, there are q^k lists oflength k from a set of size q. There are q choices for each position, regardless of the choices in other positions. By the product rule, there are q^k ways to form the k-tuple.

Definition. A permutation of a finite set S is a bijection from S to S. The word form of a permutation f of $[n]$ is the list $f(1) \ldots\ldots f(n)$ in that order. An arrangement of elements from a set S is a list of elements of S (in order). We write $n!$ (read as n factorial") to mean $\prod_{i=1}^{n} i$, with the convention that $0! = 1$.

The word form of a permutation of $[n]$ includes the full descnption of the permutation. For counting purposes we refer to the word form *as* the permutation; thus 614325 is a permutation of [6]. With this viewpoint, a permutation of $[n]$ is an arrangement of all the elements of $[n]$.

Theorem. An n-element set has $n!$ permutations (arrangements without repetition). In general, the number of arrangements of k distinct elements from a set of size n is $n(n-1) \ldots (n-k+1)$.

Proof: We count the lists of k distinct elements from a set S of size n. There is no such list when $k > n$, which agrees with the formula. We construct the lists one element at a time, $\text{speci}\Psi\dot{\text{m}}\text{g}$ the element in position $i+1$ after specifying the , elements in earlier positions.

There are n ways to choose the image of 1. For each way we do this, there are $n-1$ ways to choose the image of 2. In general, after we have chosen the first i images, avoiding them leaves $n-i$ ways to choose the next image, no matter how we made the first i choices. The rule of product yields $\prod_{i=0}^{k-1}(n-i)$ for the number of arrangements.

Often the order of elements in a list is unimportant.

Definition. A selection of k elements from $[n]$ is a k-element subset of $[n]$. The number of such selections is n choose k written as $\binom{n}{k}$.

If $k < 0$ or $k > n$, then $\binom{'n}{k} = 0$; in these cases there are no selections of k elements from $[n]$. When $0 \leq k \leq n$, we obtain a simple formula.

Theorem. For integers n, k with $0 \leq k \leq n$, $\binom{n}{k} = \frac{1}{k!}\ \prod_{i=0}^{k-1}(n-i)$.

Proof: We relate selections to arrangements. We count the arrangements of k elements from $[n]$ in two ways. Picking elements for positions as in aboue discussed Theorem yields $n(n-1) \cdot (n-k+1)$ as the number of arrangements.

Alternatively, we can select the k-element subset first and then write it in some order. Since by definition there are $\binom{n}{k}$ selections, the product rule yields $\binom{n}{k}\ k!$ for the number of arrangements.

In each case, we are counting the set of arrangements, so we conclude that

$n(n-1) \ldots (n-k+1) = \binom{n}{k} k!$. Dividing by $k!$ completes the proof. of Theorem

The numbers $\binom{n}{k}$ are called the binomial coefficients due to their appearance as coefficients in the nth power of a sum of two terms.

Theorem. (Binomial Theorem) For $n \in \mathrm{N}$, $(x+y)^n = \sum_{k=0}^{n}\ \binom{n}{k} x^k y^{n-k}$.

Proof. The proof interprets the process of multiplying out the factors in the product $(x+y)(x+y) \ldots (x+y)$. To form a term in the product, we must choose x or y from each factor. The number offactors that contribute x is some integer k in $\{0,,n\}$, and the remaining $n-k$ factors contribute y. The number of terms of the fom $x^k y^{n-k}$ is the number of ways to choose k of the factors to contnbute x. Summing over k accounts for all the terms.

Using the definition of size and the composition of bijections, it follows that finite sets A and B have the same size if and only if there is a bijection from A to B. Thus we can compute the size of a set by establishing a bijection from it to a set of known size.

Simple examples include the statements that a complete graph has $\binom{n}{2}$ edges and that therefore there are $2^{\binom{n}{2}}$ simple graphs with vertex set $[n]$. Proposition uses a bijection to count 6-cycles in the Petersen graph. Exercise uses a bijection to count graphs with vertex set $[n]$ and even vertex degrees.

Lemma. For $n \in$ N, the number of subsets of $[n]$ with even size equals
the number of subsets of $[n]$ with odd size.
.Proof. *Proof 1* (bijection). For each subset with even size, delete the element n if it appears, and add n if it does not appear. This always changes the size by 1 and produces a subset with odd size. The map is a bijection, since each odd subset containing n anses only from one even subset omitting n, and each odd subset omitting n arises on! y from one even subset containing n.
Proof 2 (binomial theorem). Setting $x = -1$ and $y = 1$ in above Theorem .yields $\sum_{k=0}^{n} \binom{n}{k}$ $(-1)^k = (-1 + 1)^n = 0$. (Note that we proved Theorem using· bijections.)
We prove a few identities involving binomial coefficients to illustrate combinatorial arguments involving bijections and the idea of counting a set in two ways. We can prove an equality by showing that both sides count the same set.

Lemma. $\binom{n}{k} = \binom{n}{n-k}$.
Proof: *Proof 1* (counting two ways). By definition, $[n]$ has $\binom{n}{k}$ subsets of size k. Another way to count selections of k elements is to count selections of $n - k$ elements to omit, and there are $\binom{n}{n-k}$ of these.
Proof 2 (bijections). The left side counts the k-element subsets of $[n]$, the ńght side counts the $n - k$-element subsets, and the operation of complementation" establishes a bijection between the two collections.
Often, "counting two ways" means grouping the elements in two ways.
Sometimes one of the counts only gives a bound on the size of the set. In this case the counting argument proves an inequality;. Here we stick to equalities.

Lemma. (The Chairperson Identity) $k\binom{n}{k} = n\binom{n-1}{k-1}$.

Proof: Each side counts the k-person committees with a designated chairperson that can be formed from a set of n people. On the left, we select the committee and then select the chair from it; on the right, we select the chair first and then fill out the$_{\,l}$est ofthe committee.

Many students see the next formula as the first application of induction, but it also is easily proved by counting a set in two ways.

Lemma. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proof: The right side is $\binom{n+1}{2}$; we can view this as counting the nontrivial intervals with endpoints in the set $\{1,, n+1\}$. On the other hand, we can group the intervals by length; there is one interval with length n, two with length $n-1$, and so on up to n intervals with length 1.

Lemma generalizes to $\sum_{i=k}^{n} \binom{j}{J} = \binom{n+1}{k+1}$. To prove this by counting in two ways, partition the set of $k+1$-element subsets of $[n+1]$ into groups so that the size of the ith. group will be $\binom{i}{k}$. Finally, a recursive computation for the binomial coefficients.

Lemma. (Pascal's Formula⟩ If $n \geq 1$, then $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Proof: We count the k-sets in $[n]$. There are $\binom{n-1}{k}$ such sets not containmg n and $\binom{n-1}{k-1}$ such sets containing n.

Given the initial conditions for $n = 0$, which are $\binom{0}{0} = 1$ and $\binom{0}{k} = 0$ for

$k \neq 0$, Pascal's Formula can be used to give inductive proofs of many statements about binomial coefficients, including above Theorems .

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