JETIR.ORG ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue
JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)
An International Scholarly Open Access, Peer-reviewed, Refereed Journal

Stability analysis of first order homogeneous Sylvester system of difference equations

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Abstract: This paper presents a criteria for the analysis of Stability, Observability and Realizability criteria for the homogeneous first order Sylvester system. Stability, Observability and Realizability for homogeneous Lyapunov system are deduced as a particular case of our results.

Keywords and Phrases : Fundamental matrix, Homogeneous first order homogeneous systems, Stability analysis.
AMS (MoS) classifications: 34B35, 34D05, 34C11

1. Introduction:

This paper presents a criteria for the existence and uniqueness of solutions to general first order difference equation

$$T(n+1) = AT(n)B \qquad (1.1)$$

where A and B are constant square matrices of order (k x k) and T is an unknown (k x k) square matrix. The general solution of homogeneous system is due to Murty, Anand and Lakshmi Prasanna [9] published in proceedings of American Mathematical society. Our aim in this paper is to study the stability analysis of (1.1) and obtain the stability criteria of the Lyapunov system.

$$T(n+1) = AT(n)A^* \quad \text{as a particular case.} \qquad (1.2)$$

Difference equations in-fact appears as a natural description of the observed evaluation phenomena because most measurements of time evolving variables are discrete and as such these equations are in their own right are important mathematical models. Apart from that difference equations also appear in the study of discretization methods for differential equations. Several results on the theory of difference equations have been obtained as natural discrete an analog of the continuous differential equations. This is essentially available in the case of boundary value problems and Lyapunov theory of stability. Nevertheless, the theory of difference equations is lot richer than the corresponding theory of differential equation. Recently Sriram Bhagavatula, Yan Wu and Murty [4, 5] obtained iterative behaviors to the solution of the Third order differential equations via fixed point methods. These results can be generalized to discrete third order systems using fixed point iterative techniques, as the theory is much more richer than the theory of differential equations.

The paper is organized as follows: Section 2 deals with preliminary notions and our main results and conclusions are presented in Section 3. It may be noted that the results established by Sriram Bhagavatula, P.Sailaja and K.N.Murty [9] presented a systematic approach on Kronecker product first order difference system in [5,6, 7]. Lakshmi, Sriram and Madhu [7] presented a systematic analysis of Linear system on Time scalar dynamical system in [1, 8].

2. Preliminaries:

We shall use in this paper of N^+ and $N^+_{n_0}$ are defined as initial sets. We shall denote a sequence $\{y_n\}$ which is the set of all values of the function y on $N^+_{n_0}$ .The Sylvester equation

$$T(n+1) = AT(n)B + F(n) \qquad (2.1.\ 1)$$

where F(n) is a (k x k) square given matrix. For every n $\in N^+_{n_0}$, the equation (2.1.1) is said to be homogeneous, if $F \equiv 0$. It may be noted that Wronskian of the matrix $K(n)$ is defined as the set of k functions $F(n)$ on $N^+_{n_0}$ as

$$K(n) = \begin{bmatrix} F_1(n) & F_2(n) & . & . & . & . & F_k(n) \\ F_1(n+1) & F_2(n+1) & . & . & . & . & F_k(n+1) \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ F_1(n+k-1) & F_2(n+k-1) & . & . & . & . & F_k(n+k-1) \end{bmatrix}$$

Let L be the linear differential operator given by

$$L(y_n) = A(n)y_n + b_n \ , \qquad (2.1.\ 2)$$

where A is a (k x k) matrix, y_n is a given (k x 1) vector and $b_n \in R^n$

The homogeneous equation associated with (2.1.2) is given by

$$L(y_n) = y_{n+1} = A(n)y_n . \qquad (2.1.\ 3)$$

Let $e_1, e_2 e_3, ..., e_k$ be the unit vectors of R^k and $y(n, n_0, e_i)$ i = 1, 2, 3,…..k the k solutions having e_i as the initial vectors.

Lemma: 2.1: Any elements of S (the space of solutions passing through y_{n_0}) can be expressed as a linear combination of $y(n, n_0, e_i)$ i = 1,2,3,…..k.

Proof: Let $y(n, n_0, c)$ be a solution of (2.1.3) satisfying $y(n_0) = C$ from the linearity of S

and from $C = \sum_i^n c_i e_j$.

It follows that $Z_n = \sum_i^n c_i\, y(n, n_0, e_i)$ satisfies (2.1.3),

and has C as initial vector. Thus from the uniqueness of initial value problems $z_n = y(n, n_0, c)$

Note that, if the columns of $K(n)$ are linearly independent and each column of $K(n)$ is solution of (2.1.3) then $\det(K(n+1)) = \det A(n). \det(K(n))$.

The matrix satisfying $K(n+1) = A(n)K(n)$ is called the Casorate matrix.

Definition 2.1.1: Given any k linear independent solutions of (2.1.3) and a vector $c \in R^k$ of arbitrary components then $K(n)C$ is called general solution of (2.1.3). Fixing the initial value condition $y(n_0) = y_{n_0}$ we get

$$y(n, n_0, y_{n_0}) = K(n) K^{-1}(n) y_{n_0} \qquad (2.\ 1.4)$$

we use the notation $\Phi(n,s) = K(n)K^{-1}(s)$ since $\Phi(n,s)$ (4) satisfies (2.1.2), It follows that $\Phi(n+1, s) = A(n)\Phi(n,s)$ .Moreover $\Phi(n,n) = I\,k$ for all $n \geq n_0$. We shall call Φ is the fundamental matrix. Now, we turn our attention to the linear system (2.1.1). Let $y(n)$ be a fundamental matrix solution of $T(n+1) = AT(n)$ and $Z^*(n)$ be a fundamental matrix solution of $T(n+1) = B^* T(n)$.

Then any solution of homogeneous equation $T(n+1) = AT(n)B$ is of the form $T(n) = Y(n)CZ^*(n)$ where C is the (k x k) square matrix and in fact a constant non-singular matrix.

For

$$(YCZ^*)(n+1) = AYCZ^*B$$

$$Y\ (n+1)CZ^*(n+1) = AY(n)CZ^*B$$

since Z is a fundamental matrix solution of $Z(n+1) = B^* Z(n)$.

It follows that $Z^*(n+1) = Z^*(n)B$. Hence $T(n) = Y(n)CZ^*(n)$ is the solution of homogeneous system $T(n+1) = AT(n)B$.

If $T(n)$ is any solution of (2.1.1) and $\overline{T(n)}$ is a particular solution of (2.1.1), then $T(n) - \overline{T(n)}$ is a solution of the homogeneous system $T(n+1) = AT(n)B$ .Thus

$$T(n) - \overline{T(n)} = Y(n)CZ^*(n)$$

(or)

$$T(n) = \overline{T(n)} + Y(n)CZ^*(n).$$

A particular solution $\overline{T(n)}$ of the system (2.1.1) is given by

$$\overline{T} = \sum_{j=n_0}^{n-1} Y(n, j)CZ^*(n, j).$$

Therefore

$$T(n) = Y(n)CZ^*(n) + \sum_{j=n_0}^{n-1} Y(n, j)CZ^*(n, j).$$

To study the Stability, Observability and Realizability criteria for the homogeneous system $T(n+1) = AT(n)B$ we need to study the behavior of solutions of the fundamental matrices $Y(n, j)$ and $Z^*(n, j)$.

3. Stability, Controllability and Observability criteria:

In this section, we establish our main results on Stability, Controllability and Realizability criteria associated with the system (2.1.1). In fact, our aim is to develop a solid foundation for a linear system theory which in fact coincides with the existing canonical system theory in the discrete case. A fascinating fact is that all the widely different disciplines of application depend on a common core of Mathematical techniques of the modern control theory [1]. In this section, we also present a set of necessary and sufficient conditions for the first order time scale discrete system (2.1.1) to be completely controllable, observable and realizable. We also present Stability criteria for linear Non-homogeneous system of the form:

$$T(n+1) = AT(n)\,B(n) + B(n)U(n), \qquad T(n_0) = T_{n_0} \tag{3.1.1}$$

$$Y(n) = C(n)T(n) + D(n)U(n)\ . \tag{3.1.2}$$

Defination3.1.1: The Linear system $T(n+1) = AT(n)\,B$ satisfying the initial condition $T(n_0) = T_{n_0}$ is said to be stable, if there exists two positive constants M_1 and M_2 such that $\|Y(n,n_0)\| \le M_1$ and $\|Z(n,n_0)\| \le M_2$ for all $n \ge n_0$

Definition 3.1.2: The linear system (3.1.2) is said to be uniformly stable, If there exist two positive constants λ and υ such that $\|T(n+1)\| \le \|T_{n_0}\| e^{-(\lambda+\upsilon)(t-t_0)}$, $\quad n \ge n_0$

Theorem 3.1.1: The linear system (3.1.2) satisfying $T(n_0) = T_{n_0}$ is uniformly exponential stable if and only if for some $\in > 0$,

$$\lim_{n\to\infty} \sum_{j=n_0}^{n-1} \| Y(n+j)\,C\,Z^*(n+j)\| \le \|T_{n_0}\|\ \upsilon\ e^{-\lambda\,(n-n_0)}, \qquad n \ge n_0 .$$

We now confine our attention to the controllability criteria of the Linear non- homogeneous difference system (3.1.1).

Definition3.1.3: The Linear non – homogeneous system (3.1.1) is said to be completely controllable, if there exist a (k x k) symmetric controllable matrix

$$W(n,0) = \sum_{j=n_0}^{n-1} Y(n_0, j)\,B(j)\,B^*(j)C\,Z^*(n_0, j) \quad \text{is non- singular,}$$

where $Y(n, n_0)$ is a fundamental matrix solution of $T(n+1) = AT(n)$ and $Z(n, n_0)$ is a fundamental matrix solution of $T(n+1) = T(n)\,B^*$ .The homogeneous Linear system (3.1.2) is said to be asymptotically stable if

$$\| Y(n,\ n_0) \| \to 0 \quad as \quad \|Z(n,\ n_0)\| \le M \qquad \text{for all } n \ge n_0$$

(or)

$$\| Z(n,\ n_0) \| \to 0 \quad as \quad \|Y(n,\ n_0)\| \le m \qquad \text{for all } n \ge n_0 .$$

Definition 3.1.4: The non- homogeneous system (3.1.1) is said to be completely controllable if for any initial state $T(n_0)=Tn_0$ and given final state $T(n)$ there exist a finite time $n \geq n_0$ and a control $U(n)$, $n_0 \leq n \leq n_f$ such that $T(n_f) = T_f$.

Definition 3.1.5: The non –homogeneous system (3.1.1) is said to be completely observable over the interval $[\ n_0 \ \ N\ ]$ if the knowledge of the rule of the base of input $\hat{U}$ output $\hat{Y}$ over $[\ n_0 \ \ N\ ]$ satisfies to determine the rule base of the initial system $\hat{T}_{n_0}$.

Theorem 3.1.2: The non-homogeneous system (3.1.1) is completely controllable if the $k^2 x k^2$ symmetric controllable matrix

$$W(0,N)=\sum_{j=n_0}^{n-1} Y(n-j-1)\,Z^*(n-j-1)U_j\,U_j^*\,Z(n-j-1)Y^*(n-j-1) \qquad (3.1.3)$$

(where * indicates the complex conjugate) is non- singular.

Proof: Suppose that the controllability matrix $W(0,N)$ is non-singular. Then $W^{-1}(0,N)$ exists and therefore multiply (3.1.3) by $W^{-1}(0,N)\Phi(N)Z(N)$,

we get

$$Y(n)\ Z^*(n)T(n_0)=\sum_{j=0}^{n-1} Y(n-j-1)\,Z^*(n-j-1)U_jU_j^*Z(n-j-1)Y^*(n-j-1).w^{-1}(n_0,N)Y(n-j-1)$$
$$Z^*(n-j-1)T(n_0)\,Z(n-j-1)Y^*(n-j-1).$$

Now, our problem is to find the control $\hat{U}(n)$ such that

$$\hat{T}(n) = \hat{T}_f = Y(n-j-1)Z^*(n-j-1)\hat{T}_0\,Z(n-j-1)\,Y^*(n-j-1)$$
$$+\sum_{j=0}^{N-1} Y(n-j-1)\,Z^*(n-j-1)U_jU_j^*Z(n-j-1)Y^*(n-j-1)$$

now, $\hat{U}(N)$ can be expressed as

$$\hat{U}(N) == \frac{1}{N}\sum\left\{(U_j,U_j^*)^{-1}Y(n-j-1)Z^*(n-j-1)U_jU_j^*Z(n-j-1)Y^*(n-j-1)\hat{T}_0\right\}.$$

Now, two cases arise

Case 1: When $\hat{T}(N)=\hat{T}_{nf}$ then, the corresponding control $\hat{U}(n)$.

Case 2: When $\hat{T}(N)=\{\hat{T}_1(n), \hat{T}_2(n), \hat{T}_3(n),........., \hat{T}_{k^2}(n)\}$, equation (3.1.3) gives the control $\hat{U}(n)$ and the proof of the theorem is complete.

Theorem 3.1.3: The non- homogeneous control system (3.1.1) is completely observable over the interval $[\ n_0, N\ ]$ if and only if the symmetric $(k\ x\ k)$ observability matrix.

$$M(n_0, n_f) = \sum_{j=n_0}^{n-1} Y(n-j-1)\, C^*(n)\, C(n) Z^*(n-j-1) \text{ is non singular.}$$

Proof: First suppose that the system (3.1.1) is completely observable. Then there exist a non – zero matrix $T(n_0)$ such that $M(n_0, n_f)T(n_0) = 0$ then, clearly $T^*_{n_0}\ M(n,\ n_f)T_{n_0} = 0$, hence $C(n)Y(n, n_0)T_{n_0} Z^*(n, n_0) = 0$.

Thus, $T(n_0) = T_{n_0} + X_\alpha$ yields the same – input response for the system with $T(n_0) = T_{n_0}$

and the system is not observable on $[n_0, n_f\ ]$, a contradiction.

Conversely, suppose the Gramian matrix $M(n_0, n_f)$ is invertible. Then the solution expression $T(n) = C(n)Y(n, n_0)T_{n_0}\ Z^*(n, n_0)$ is multiplied by $Y^*(n, n_0)T_{n_0}\ C^*(n)$, and submitting to get

$$\sum_{j=n_0}^{n-1} Y(n, n-j-1)\, C^*(n)\, C(n) Z^*(n,\ n-j) = M(n\ , n_f)\, T_{n_0} .$$

The left-hand side of the above expression is determined by $T(n)$ for $n \in [\ n_0, n_f\ ]$and the equation is linear algebraic equation in T_{n_0} . Since $M(n_0, n_f)$is non-singular it follows that T_{n_0} is determined uniquely and hence to state equation is observable. This is there for all $n \in [\ n_0, n_f\ ]$ it follows that the state equation is completely observable and to proof of the theorem is complete.

Now, we confine our attention to the two first order linear differential equations of n^{th} order and study stability analysis.

For, let

$$Lu(n) = u^{(k)} + p_1 u^{(k-1)} + ... + p_n u = f(n) \tag{3.1.4}$$

and

$$Lv(n) = v^{(k)} + q_1 v^{(k-1)} + ... + q_n v = g(n) . \tag{3.1.5}$$

If $u_{1,} u_2 ,........., u_k$ be k linearly independent solutions of $Lu(n) = 0$, Then $K\ (u(n))$ is a fundamental matrix of the companion linear system and similarly , if $v_{1,} v_2 ,........., v_k$ are k linearly independent solutions of $Lv(n) = 0$, then $K\ (v(n))$ is a fundamental matrix of the companion linear system then any solution of $LT(n) = AT(n)B$

is of the from $T(n) = K(u(n))\, CK^*(v(n))$,

where C is a constant (kxk) square matrix. The controllability and observability criteria established above can be studies for the linear system $T(n+1) = AT(n)B$.

In order to avoid monotony, we even omit staying those results.

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