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APPROXIMATE BAYES ESTIMATE FOR THE PARAMETERS OF GENERALIZED COMPOUND RAYLEIGH DISTRIBUTION UNDER LINEX LOSS FUNCTION

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Abstract- The Generalized Compound Rayleigh Distribution (GCRD) is a special case of the three parameter Burr type XII distribution. This uni-modal hazard function is generalized and a flexible parametric model is thus constructed, which embeds the compound Rayleigh model, by adding shape parameter. The main objective of this paper is determine the best estimator for the three parameter Generalized Compound Rayleigh Distribution assuming all the parameters unknown. The methods under consideration are Maximum Likelihood Estimation and Bayesian Estimation underline Loss Function. Lindley Approximation is use to obtain the Approximate Bayes Estimate of GCRD assuming all the parameters unknown under LINEX LOSS FUNCTION. We have studied the sensitivity of the Approximate Bayes Estimate of the model and presented a numerical study to illustrate the above technique on generated observations. The comparison is done by R-programming.

Keywords- Maximum Likelihood Estimation, Bayesian Approximation, Linex Loss Function, Lindley Approximation

1. Introduction

Mostert , Roux, and Bekker (1999) considered a gamma mixture of Rayleigh distribution and obtained the compound Rayleigh model with unimodal hazard function.
The Generalized Compound Rayleigh Distribution is a special case of three- parameter Burr type XII distribution with probability density function,

$$f(x;\alpha,\beta,\gamma) = \alpha\gamma\beta^{\gamma}x^{\alpha-1}(\beta + x^{\alpha})^{-(\gamma+1)} \qquad x,\alpha,\beta,\gamma > 0 \qquad (1.1)$$

With Probability Distribution Function

$$F(x) = 1 - (1 - \beta x^{\alpha})^{-\gamma} \qquad x,\alpha,\beta,\gamma > 0 \qquad (1.2)$$

Reliability function is

$$R(t) = \left(\frac{\beta}{\beta+t^{\alpha}}\right)^{\gamma}$$

Hazard rate function

$$H(t) = \alpha\gamma\frac{t^{\alpha-1}}{\beta+t^{\alpha}}$$

The Generalized compound Rayleigh model includes various well-known pdfs, namely

(i) Beta-Prime pdf (Patil, et al., 1984), if $\alpha = \beta = 1$
(ii) $\alpha = 1$

(iii) Burr XII pdf(Burr,1942), if $\beta = 1$
Compound Rayleigh pdf (Siddiqui &Weiss,1963), if $\alpha = 2$

Ferguson (1967), Zellner & Geisel (1968), Rojo (1987), Aitchision & Dunsmore (1975) and Berger (1980) have considered the linear asymmetric loss function. Varian (1975) introduced the following convex loss function known as LINEX. (Linear Exponential) Loss Function i.e. given as;

$L(\Delta) = be^{a\Delta} - c\Delta - b\,; a, c \neq 0, b > 0$

Where $\Delta = \hat{\theta} - \theta$. It is clear that L(0) = 0 and the minimum occurs when ab=c, therefore , $L(\Delta)$ can be written as

$$L(\Delta) = b[e^{a\Delta} - a\Delta - 1\,], a \neq 0, b > 0; \qquad (1.3)$$

Where a and b are the parameters of the loss function may be defined as shape and scale respectively. The loss function has been considered by Zellner (1986) , Rojo (1987) , Basu and Ebrahimi (1991) considered the $L(\Delta)$ as

$$L(\Delta) = b[e^{a\Delta} - a\Delta - 1\,],\ a \neq 0, b > 0 \qquad (1.4)$$

Where, $\Delta = \frac{\hat{\theta}}{\theta} - 1$

and studied the Bayesian estimates under this linex loss function for exponential lifetime distribution. This loss function is suitable for the situation where overestimation of θ is more costly than its underestimation .This loss function $L(\Delta)$ havethe following properties:

For a=1, the function is quite asymmetric about zero with overestimation being more costly than underestimation.

For $a < 0$, $L(\Delta)$ rises exponentially when $\Delta < 0$ (underestimation) and almost linearly when $\Delta > 0$ (overestimation); and

For small values of $|a|$, $L(\Delta)$ is almost symmetric and not from a squared error losses function, Indeed, on expanding

$$e^{a\Delta} \simeq 1 + a\Delta + \frac{a^2\Delta^2}{2} \quad \text{or} \quad L(\Delta) \simeq \frac{a^2\Delta^2}{2}; \qquad (1.5)$$

is a squared error loss function . Thus for small values of $|a|$, optimal estimates are not for different from those obtained with a squared error loss function.

2. The Estimators

Let $x_1 \leq x_2 \leq \ldots\ldots\ldots \leq x_n$ be the n failures in complete sample case. The likelihood function is given by;

$$L(\underline{x}\,|\alpha,\beta,\gamma) = \prod_{j=1}^{n} f\,(x_j,\alpha,\beta,\gamma)$$

$$= \alpha^n\gamma^n\beta^{n\gamma} \prod_{j=1}^{n} x_j{}^{\alpha-1} \prod_{j=1}^{n}(\beta + x_j{}^{\alpha})^{-(\gamma+1)}$$

$$L(\underline{x}\,|\alpha,\beta,\gamma) = (\alpha\gamma)^n U e^{-\gamma T} \qquad (2.1)$$

where

$$T = \sum_{j=1}^{n} log\left[1 + \frac{x_j{}^{-\alpha}}{\beta}\right] \qquad \text{and} \qquad U = \prod_{j=1}^{n} \frac{x_j{}^{\alpha-1}}{(\beta + x_j{}^{\alpha})}$$

from equation(2.1) the log likelihood function is

$$Log\ L = n\ log\alpha + nlog\gamma + n\gamma\ log\beta + (\alpha - 1)\sum_{j=1}^{n} logx_j$$

$$-(\gamma + 1)\sum_{j=1}^{n}(\beta + x_j{}^{\alpha}) \qquad (2.2)$$

and differentiation of equation(2.2) with respect to α, β and γ yields respectively we get

$$\frac{\partial \, Log\, L}{d\alpha} = \frac{n}{\alpha} + \sum_{j=1}^{n} log x_j - \sum_{j=1}^{n} \frac{x_j{}^{\alpha} log x_j}{\beta + x_j{}^{\alpha}} - \gamma \sum_{j=1}^{n} \frac{x_j{}^{\alpha} log x_j}{\beta + x_j{}^{\alpha}} \quad (2.3)$$

$$\frac{\partial \, Log\, L}{\partial\beta} = -\sum_{j=1}^{n} \frac{1}{\beta + x_j{}^{\alpha}} + \gamma \sum_{j=1}^{n} \frac{x_j{}^{\alpha}}{\beta(\beta + x_j{}^{\alpha})} \quad (2.4)$$

$$\frac{\partial \, Log\, L}{\partial\gamma} = \frac{n}{\gamma} + n \log\beta - \sum_{j=1}^{n} log\left(1 + \frac{x_j{}^{\alpha}}{\beta}\right) - n \, log \, \beta$$

$$= \frac{n}{\gamma} - \sum_{j=1}^{n} log\left(1 + \frac{x_j{}^{\alpha}}{\beta}\right) \quad (2.5)$$

setting the expressions for the derivatives in 8 equal to zero and solving α, β and γ yield. The maximum likelihood estimators (MLE) of the parameters namely $\hat{\alpha}_{MLE}$, $\hat{\beta}_{MLE}$ and $\hat{\gamma}_{MLE}$.

However, no closed form solutions exist in this case the elimination of γ in $\frac{\partial \, Log\, L}{\partial\beta}$ and $\frac{\partial \, Log\, L}{\partial\alpha}$ and in $\frac{\partial \, Log\, L}{\partial\alpha}$ and $\frac{\partial \, Log\, L}{\partial\gamma}$ yield a set of equations in terms of β and α.

$$\frac{\sum_{j=1}^{n} \frac{1}{\beta + x_j{}^{\alpha}}}{\sum_{j=1}^{n} \frac{x_j{}^{\alpha}}{\beta + x_j{}^{\alpha}}} - \frac{n}{\sum_{j=1}^{n} log\left[1 + \frac{x_j{}^{\alpha}}{\beta}\right]} = 0 \quad (2.6)$$

and

$$\frac{n}{\alpha} + \sum_{j=1}^{n} \log x_j - \Sigma \frac{x_j{}^{\alpha} log x_i}{(\beta + x_j{}^{\alpha})} - \frac{n \sum_{j=1}^{n} \frac{x_j{}^{\alpha} log x_i}{(\beta + x_j{}^{\alpha})}}{\sum_{j=1}^{n} log\left[1 + \frac{x_j{}^{\alpha}}{\beta}\right]} = 0 \quad (2.7)$$

respectively. Applying the Newton-Raphson method $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$ can be derived and then from them $\hat{\gamma}_{MLE}$ can be obtained.

Bayes estimate for γ with known parameter α, β and γ

If $\hat{\alpha}$ and $\hat{\beta}$ is known we assume $\gamma(a, b)$ as conjugate prior for γ as

$$g(\gamma|\underline{x}) = \frac{b^a}{\Gamma a} \gamma^{a-1} e^{-\gamma b} \; ; \quad (a, b) > 0, \gamma > 0 \quad (2.8)$$

combining the likelihood function equation(2.1) and probability density equation(1.1) we obtain the posterior density of γ in the form of

$$h(\gamma|\underline{x}) = \frac{\alpha^n \gamma^n \beta^{\gamma n} \prod_{j=1}^{n} x_j{}^{\alpha-1} \prod_{j=1}^{n} (\beta + x_j{}^{\alpha})^{-(\gamma+1)} \frac{b^a}{\Gamma a} \gamma^{a-1} e^{-b\gamma}}{\int_0^{\infty} \alpha^n \gamma^n \beta^{\gamma n} \prod_{j=1}^{n} x_j{}^{\alpha-1} \prod_{j=1}^{n} (\beta + x_j{}^{\alpha})^{-(\gamma+1)} \frac{b^a}{\Gamma a} \gamma^{a-1} e^{-b\gamma} \, d\gamma} \quad (2.9)$$

$$h(\gamma|\underline{x}) = \frac{\gamma^{n+a-1} e^{-\gamma(b+T)}}{\int_0^{\infty} \gamma^{n+a-1} e^{-\gamma(b+T)} d\gamma}$$

Assuming;

$$\sum_{j=1}^{n} \log\left(1 + \frac{x_j{}^{\alpha}}{\beta}\right) = T$$

$$h(\gamma|\underline{x}) = \frac{\gamma^{n+a-1} e^{-\gamma(b+T)} (b+T)^{n+a}}{\Gamma(n+a)} \quad (2.10)$$

3.Bayes Estimator under Linex Loss Function

Let the loss function L(Δ) is

$$L(\Delta) = e^{-k\Delta} - k\Delta - 1; \qquad k = 0 \qquad (3.1)$$

Where

$$\Delta = (\hat{u} - u) \qquad where \;\; u = u(\alpha, \beta, \gamma)$$

Now

$$\mathrm{E}\big(\mathrm{L}(\Delta)\big) = \mathrm{E}\big(e^{k(\hat{u}-u)} - \mathrm{k}(\hat{u} - u) - 1\big)$$

$$= e^{k\hat{u}} Eu\big(e^{-k\hat{u}}\big) - k\big(\hat{u} - Eu(u)\big) - 1$$

$$\Rightarrow \hat{u}_{ABL} = -\frac{1}{k}\log E_{\mu}\left(e^{-k\hat{u}}\right) \qquad (3.2)$$

The Bayes estimator under asymmetric loss is given by

$$\Rightarrow \hat{\gamma}_{ABL} = -\frac{1}{k}\log\big(E_h(e^{-k\gamma})\big)$$

Now

$$E_h(e^{-k\gamma}) = \int_0^{\infty} \frac{e^{-k\gamma}\gamma^{(n+a-1)} e^{-\gamma(b+T)}(b+T)^{(n+a)} d\gamma}{\Gamma(n+a)} \qquad (3.3)$$

$$= \hat{\gamma}_{BL} = \frac{(n+a)}{k} log\left(1 + \frac{k}{(b+T)}\right)$$

4 Approximate Bayes Estimators with unknown α, β and γ

Joint prior density α, β, γ is given by

$$\mathrm{G}\ (\alpha, \beta, \gamma) = g_1(\alpha) g_2(\beta) g_3(\gamma|\beta)$$

$$= \frac{c}{\delta \Gamma \xi} \beta^{-\xi} \gamma^{\xi - 1} exp\left[-\left(\frac{\gamma}{\beta} + \frac{\beta}{\delta}\right)\right] \qquad (4.1)$$

where

$$g_1(\alpha) = c \qquad (4.2)$$

$$g_2(\beta) = \frac{1}{\delta} e^{-\frac{\beta}{\delta}} \qquad (4.3)$$

$$g_3(\gamma) = \frac{1}{\Gamma \xi} \beta^{-\xi} \gamma^{\xi+1} e^{-\frac{1\gamma}{\beta}} \qquad (4.4)$$

The Joint posterior with likelihood equation (2.10) and joint prior equation (4.1)

$$h^*(\alpha, \beta, \gamma) = \frac{\beta^{-\xi}\gamma^{\xi-1} exp\left[-\left(\frac{\gamma}{\beta} + \frac{\beta}{\delta}\right)\right] . L(\underline{x}|\alpha, \beta, \gamma)}{\int_\alpha \int_\beta \int_\gamma \beta^{-\xi}\gamma^{\xi-1} exp\left[-\left(\frac{\gamma}{\beta} + \frac{\beta}{\delta}\right)\right] . L(\underline{x}|\alpha, \beta, \gamma) d\alpha d\beta d\gamma} \qquad (4.5)$$

The approximate Bayes estimators are evaluated as

$$\cup\ (\Theta) = \cup\ (\alpha, \beta, \gamma)$$

$$\hat{\cup}_{BS} = E\left(\cup\,|\underline{x}\right) = \frac{\int_\alpha \int_\beta \int_\gamma \cup(\alpha, \beta, \gamma) G^*(\alpha, \beta, \gamma) d\alpha d\beta d\gamma}{\int_\alpha \int_\beta \int_\gamma G^*(\alpha, \beta, \gamma) d\alpha d\beta d\gamma} \qquad (4.6)$$

Lindley Approximation Procedures

The ratio of integrals in equation (4.6) does not seem to take a closed form so we must consider the Lindley approximation procedure as

$$E\big(\mu(\theta, p)\big|\underline{x}\big) = \frac{\int \mu(\theta) . e^{(l(\theta) + \rho(\theta))} d\theta}{\int e^{(l(\theta) + \rho(\theta))} . d\theta} \qquad (4.7)$$

Lindley developed approximate procedure for evaluation of posterior expectation of $\mu(\theta)$. Several other authors have used this technique to obtain Bayes estimators (see Sinha(1986), Sinha and sloan(1988),Soliman(2001)).The posterior expectation of Lindley approximation procedure to evaluate of $\mu(\theta)$ in equation (4.6 and 4.6) under LLF, where where $\rho(\theta) = \log g(\theta)$, and $g(\theta)$ is an arbitrary function of θ and $l(\theta)$ is the logarithm likelihood function (Lindley (1980)).
The modified form of equation (4.7) is given by

$$E(\cup(\alpha,\beta,\gamma|\underline{x}) = \cup(\Theta) + \frac{1}{2}(A\cup_1\sigma_{11} + \cup_2\sigma_{12} + \cup_3\sigma_{13}) + B(\cup_1\sigma_{21} + \cup_2\sigma_{22} + \cup_3\sigma_{23}) + P(\cup_1\sigma_{31} + \cup_2\sigma_{32} + \cup_3\sigma_{33})] + (\cup_1 a_1 + \cup_2 a_2 + \cup_3 a_3 + a_4 + a_5) + 0\left(\frac{1}{n^2}\right) \quad (4.8)$$

Evaluated at MLE = $(\hat{\alpha},\hat{\beta},\hat{\gamma})$ where

$$a_1 = \rho_1\sigma_{11} + \rho_2\sigma_{12} + \rho_3\sigma_{13} \quad (4.8)$$

$$a_2 = \rho_1\sigma_{21} + \rho_2\sigma_{22} + \rho_3\sigma_{23} \quad (4.9)$$

$$a_3 = \rho_1\sigma_{31} + \rho_2\sigma_{32} + \rho_3\sigma_{33} \quad (4.10)$$

$$a_4 = \cup_{12}\sigma_{12} + \cup_{13}\sigma_{13} + \cup_{23}\sigma_{23} \quad (4.11)$$

$$a_5 = \frac{1}{2}(\cup_{11}\sigma_{11} + \cup_{22}\sigma_{22} + \cup_{33}\sigma_{33}) \quad (4.12)$$

and

$$A = [\sigma_{11}l_{111} + \sigma_{12}l_{121} + 2\sigma_{13}l_{131} + 2\sigma_{23}l_{231} + \sigma_{22}l_{221} + \sigma_{33}l_{331}] \quad (4.13)$$

$$B = [\sigma_{11}l_{112} + 2\sigma_{12}l_{122} + 2\sigma_{13}l_{132} + 2\sigma_{23}l_{232} + \sigma_{22}l_{222} + \sigma_{33}l_{332}] \quad (4.14)$$

$$P = [\sigma_{11}l_{113} + 2\sigma_{13}l_{133} + 2\sigma_{12}l_{123} + 2\sigma_{23}l_{233} + \sigma_{22}l_{223} + \sigma_{33}l_{333}] \quad (4.15)$$

To apply Lindley approximation on equation (4.8) we first obtain

$$\sigma_{ij} = [-l_{ijk}]^{-1} i,j,k = 1,2,3$$

$$L = \alpha^n\gamma^n\beta^{n\gamma}\prod_{j=1}^{n} x_j^{\alpha-1}\prod_{j=1}^{n}(\beta + x_j{}^{\alpha})^{-(\gamma+1)}; (x,\beta,\gamma > 0)$$

$$Log\ L = n\ log\alpha + nlog\gamma + n\gamma log\beta + (\alpha - 1)\sum_{j=1}^{n}\log x_j - (\gamma+1)\sum_{j=1}^{n}\log(\beta + x_j{}^{\alpha})$$

Now differentiating log likelihood function with respect to α,β,γ we get

$$l_1 = \frac{n}{\alpha} + \sum_{j=1}^{n}\log x_j - (\gamma+1)\omega_{11} \quad \text{where} \quad \omega_{11} = \sum_{j=1}^{n}\frac{x_j{}^{\alpha}log x_j}{\beta + x_j{}^{\alpha}} \quad (4.16)$$

$$l_2 = \frac{n\gamma}{\beta} - (\gamma+1)\delta_{11} \quad \text{where} \quad \delta_{11} = \sum_{j=1}^{n}\frac{1}{\beta + x_j{}^{\alpha}} \quad (4.17)$$

$$l_3 = \frac{n}{\gamma} + n\log\beta - q_{11} \quad \text{where} \quad q_{11} = \sum_{j=1}^{n}\log(\beta + x_j{}^{\alpha}) \quad (4.18)$$

$$l_{11} = \frac{-n}{\alpha^2} - \beta(\gamma+1)\omega_{122} \quad \text{where} \quad \omega_{122} = \sum_{j=1}^{n}\frac{x_j{}^{\alpha}(\log x_j)^2}{(\beta + x_j{}^{\alpha})^2} \quad (4.19)$$

$$l_{22} = \frac{-n\gamma}{\beta^2} - (\gamma+1)\delta_{12} \quad \text{where} \quad \delta_{12} = \sum_{j=1}^{n}\frac{1}{(\beta + x_j{}^{\alpha})^2} \quad (4.20)$$

$$l_{33} = \frac{\partial^2 L}{\partial\gamma^2} = \frac{-n}{\gamma^2} \quad (4.21)$$

$$l_{12} = (\gamma + 1)\omega_{14} \qquad \text{where} \qquad \omega_{14} = \Sigma \frac{x_j^{\alpha} \log x_j}{(\beta + x_j)^2} \tag{4.22}$$

$$l_{21} = (\gamma + 1)\omega_{14} \tag{4.23}$$

$$l_{13} = -\sum_{j=1}^{n} \frac{x_j^{\alpha} \log x_j}{\beta + x_j^{\alpha}} = -\omega_{11} = l_{31} \tag{4.24}$$

$$l_{23} = \frac{n}{\beta} - \sum_{j=1}^{n} \frac{1}{\beta + x_j^{\alpha}} = \frac{n}{\beta} - \delta_{11} = l_{32} \tag{4.25}$$

$$l_{111} = \frac{2n}{\alpha^3} + (\gamma + 1)\beta \;\; \omega_{133}, \qquad \text{where} \quad \omega_{133} = \sum_{j=1}^{n} \frac{x_j^{\alpha} (\log x_j)^3 (\beta - x_j^{\alpha})}{(\beta + x_j^{\alpha})^3} \tag{4.26}$$

$$l_{222} = \frac{2n\gamma}{\beta^3} - 2(\gamma + 1) \;\; \delta_{13} \qquad \text{where} \qquad \delta_{13} = \sum_{j=1}^{n} \frac{1}{(\beta + x_j^{\alpha})^3} \tag{4.27}$$

$$l_{333} = \frac{\partial^3 L}{\partial \alpha^3} = \frac{2n}{\gamma^3} \tag{4.28}$$

$$l_{112} = -(\gamma + 1)\; \omega_{123} = l_{121} \;\; \text{where} \quad \omega_{123} = \sum_{j=1}^{n} \frac{x_j^{\alpha} (\log x_j)^2 (\beta - x_j^{\alpha})}{(\beta + x_j^{\alpha})^3} \tag{4.29}$$

$$l_{113} = -\beta \sum_{j=1}^{n} \frac{(\log x_j)^2 x_j^{\alpha}}{(\beta + x_j^{\alpha})^2} = -\beta\; \omega_{122} = l_{131} \tag{4.30}$$

$$l_{221} = -2(\gamma + 1)\omega_{113} = l_{212} \qquad \text{where} \qquad \omega_{113} = \sum_{j=1}^{n} \frac{x_j^{\alpha} \log x_j}{(\beta + x_j^{\alpha})^3} \tag{4.31}$$

$$l_{223} = \frac{-n}{\beta^2} + \delta_{12} = l_{232} \tag{4.32}$$

$$l_{331} = 0 = l_{313} \tag{4.33}$$

$$l_{332} = 0 = l_{323} \tag{4.34}$$

$$l_{231} = \frac{\partial}{\partial \beta}\left(\frac{\partial^2 L}{\partial \gamma \partial \alpha}\right) = 0 = l_{213} \tag{4.35}$$

$$l_{122} = \frac{\partial}{\partial \alpha}\left(\frac{\partial^2 L}{\partial \beta^2}\right) = -2(\gamma + 1)\omega_{113} \tag{4.36}$$

$$l_{132} = \frac{\partial}{\partial \alpha}\left(\frac{\partial^2 L}{\partial \gamma \partial \beta}\right) = \sum_{j=1}^{n} \frac{x_j^{\alpha} \log x_j}{(\beta + x_j^{\alpha})^2} = \omega_{112} = l_{123} \tag{4.37}$$

$$l_{133} = \frac{\partial}{\partial \alpha}\left(\frac{\partial^2 L}{\partial \gamma^2}\right) = 0 \tag{4.38}$$

$$l_{233} = \frac{\partial}{\partial \beta}\left(\frac{\partial^2 L}{\partial \gamma^2}\right) = 0 \tag{4.39}$$

The matrix of derivatives is given as

$$[-l_{ijk}] = -\begin{bmatrix} l_{111} & l_{112} & l_{113} \\ l_{221} & l_{222} & l_{223} \\ l_{331} & l_{332} & l_{333} \end{bmatrix}$$

$$[-l_{ijk}] = \begin{bmatrix} \frac{2n}{\alpha^3} + (\gamma + 1)\beta\omega_{133} & -(\gamma + 1)\omega_{123} & -\beta\omega_{122} \\ -2(\gamma + 1)\omega_{113} & \frac{2n\gamma}{\beta^3} - 2(\gamma + 1)\delta_{13} & -\frac{n}{\beta^2} + \delta_{12} \\ 0 & 0 & \frac{2n}{\gamma^3} \end{bmatrix}$$

$$[-l_{ijk}] = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$Determinant\ of - [l_{ijk}]$

D= $-M_{33}\{M_{11}M_{22} - M_{12}M_{21}\}$

$$[-l_{ijk}]^{-1} = \frac{(Adjoint\ of [-l_{ijk}])'}{[-l_{ijk}]}$$

$$[-l_{ijk}]^{-1} = \begin{bmatrix} -\frac{M_{22}M_{33}}{D} & \frac{M_{12}M_{33}}{D} & \frac{M_{22}M_{13} - M_{12}M_{23}}{D} \\ \frac{M_{21}M_{33}}{D} & -\frac{M_{11}M_{33}}{D} & \frac{M_{11}M_{23} - M_{21}M_{13}}{D} \\ 0 & 0 & \frac{M_{11}M_{22} - M_{12}M_{21}}{D} \end{bmatrix}$$

$$[-l_{ijk}]^{-1} = \begin{bmatrix} Y_{11/D} & Y_{12/D} & Y_{13/D} \\ Y_{21/D} & Y_{22/D} & Y_{23/D} \\ 0 & 0 & Y_{33/D} \end{bmatrix}$$

$$[-l_{ijk}]^{-1} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

5. Approximate Bayes Estimator under Linex Loss function

$$\hat{U}_{ABL} = -\frac{1}{k} log(E_u(e^{-ku}))$$

where

$$E_u(e^{-ku}|\underline{x}) = \frac{\int_\alpha \int_\beta \int_\gamma e^{-ku} G^*(\alpha,\beta,\gamma)\partial\alpha\partial\beta\partial\gamma}{\int_\alpha \int_\beta \int_\gamma G^*(\alpha,\beta,\gamma)\partial\alpha\partial\beta\partial\gamma} \qquad (5.1)$$

The above equation (5.1) is evaluated by method of Lindley approximation, whose simplified form is equation(4.8)

Special Cases :–

$U(\alpha, \beta, \gamma) = U$

1. Approximate Bayes Estimate of α

$U(\alpha, \beta, \gamma) = U\ e^{-k\alpha}$

$U_1 = \frac{\partial}{\partial\alpha}(e^{-k\alpha}) = -ke^{-k\alpha};\quad U_{11} = \frac{\partial}{\partial\alpha}(-e^{-k\alpha}) + k^2e^{-k\alpha},\quad U_{12} = U_{13} = 0$

$U_2 = U_{21} = U_{22} = U_{23} = 0$

$U_3 = U_{31} = U_{32} = U_{33} = 0$

$E(e^{-k\alpha}) = e^{-k\alpha} + \varphi_1 + \varphi_2$

$\varphi_1 = U_1\, a_1 + U_2\, a_2 + U_3\, a_3 + a_4 + a_5$

$$\varphi_1 = -ke^{-k\alpha}\left(-\frac{1}{2D}Y_{11}k + \left(\frac{-\xi}{\beta} + \frac{\gamma}{\beta^2} - \frac{1}{\delta}\right)\frac{Y_{12}}{D} + \left(\frac{\xi - 1}{\gamma} + \frac{-1}{\beta}\right)\frac{Y_{13}}{D}\right)$$

$$\varphi_2 = \frac{1}{2}[(A\sigma_{11} + B\sigma_{21} + P\sigma_{31})\cup_1 + (A\sigma_{12} + B\sigma_{22} + P\sigma_{32})\cup_2 + (A\sigma_{13} + B\sigma_{23} + P\sigma_{33})\cup_3]$$

$$\varphi_2 = -\frac{1}{2}[(A\sigma_{11} + B\sigma_{21})ke^{-k\alpha}]$$

$$E(e^{-k\alpha}) = e^{-k\alpha} - ke^{-k\alpha}\left(\frac{-\xi}{\beta} + \frac{\gamma}{\beta^2} - \frac{1}{\delta}\right)\frac{Y_{12}}{D} - \frac{1}{2D}Y_{11}k\left(\frac{\xi - 1}{\gamma} + \frac{1}{\beta}\right)\frac{Y_{13}}{D} - ke^{-k\alpha}\left(\frac{A\sigma_{11} + B\sigma_{21}}{2}\right)$$

$$E(e^{-k\alpha}) = e^{-k\alpha}\varphi_3$$

$$\hat{\alpha}_{ABL} = -\frac{1}{k}log[E_u(e^{-k\alpha})]$$

$$\hat{\alpha}_{ABL} = \alpha - \frac{1}{k}log\varphi_3$$

$$\Rightarrow -\frac{1}{k} = k'$$

$$\hat{\alpha}_{ABL} = \alpha + k'log\varphi_3 \text{ ; at } (\hat{\alpha}_{ML,}\hat{\beta}_{ML,}\hat{\gamma}_{ML}) \quad (5.2)$$

where

$$\varphi_3 = \left[1 - k\left(\frac{-\xi}{\beta} + \frac{\gamma}{\beta^2} - \frac{1}{\delta}\right)\frac{Y_{12}}{D} - \frac{1}{2D}Y_{11}k + \left(\frac{\xi-1}{\gamma} + \frac{1}{\beta}\right)\frac{Y_{13}}{D} + \frac{k}{2}\left\{\left(\frac{1}{D}Y_{11}\left(\frac{2n}{\alpha^3} - (\gamma + 1)\omega_{133}\right) + 2\frac{Y_{12}}{D}(\gamma + 1)\omega_{123} - 2\frac{Y_{13}}{D}\beta\omega_{122} - 2\frac{Y_{22}}{D}(\gamma + 1)\omega_{113}\right)\sigma_{11} + \left(-\frac{Y_{11}}{D}(\gamma + 1)\omega_{123} - 4\frac{Y_{12}}{D}(\gamma + 1)\omega_{113} + 2\frac{Y_{13}}{D}\omega_{112} + \frac{2Y_{23}}{D}\left(\frac{-n}{\beta^2} + \delta_{12}\right) + \frac{Y_{22}}{D}\left(\frac{2n\gamma}{\beta^3} - 2(\gamma + 1)\delta_{13}\right)\right)\sigma_{21}\right\}\right] \quad (5.3)$$

2. Approximate Bayes Estimate of β

$$\cup(\alpha, \beta, \gamma) = \cup\ e^{-k\beta}$$

$$\cup_2 = \frac{\partial}{\partial\beta}(e^{-k\beta}) = -ke^{-k\beta}\ ; \quad \cup_{22} = \frac{\partial}{\partial\beta}(-e^{-k\beta})k^2e^{-k\beta}, \quad \cup_{21} = \cup_{23} = 0$$

$$\cup_1 = \cup_{11} = \cup_{12} = \cup_{13} = 0$$

$$\cup_3 = \cup_{31} = \cup_{32} = \cup_{33} = 0$$

$$E(e^{-k\beta}) = \cup + \varphi_1 + \varphi_2$$

$$\varphi_1 = \cup_1 a_1 + \cup_2 a_2 + \cup_3 a_3 + a_4 + a_5$$

$$\varphi_1 = -e^{-k\beta}\left(k\left(\frac{-\xi}{\beta} + \frac{\gamma}{\beta^2} - \frac{1}{\delta}\right)\frac{Y_{22}}{D} + k\left(\frac{\xi - 1}{\gamma} - \frac{1}{\beta}\right)\frac{Y_{23}}{D} - k^2\frac{Y_{22}}{2D}\right)$$

$$\varphi_2 = \frac{1}{2}[(A\sigma_{11} + B\sigma_{21} + P\sigma_{31})\cup_1 + (A\sigma_{12} + B\sigma_{22} + P\sigma_{32}) + \cup_2 + (A\sigma_{13} + B\sigma_{23} + P\sigma_{33})\cup_3]$$

$$= \frac{1}{2}[(A\sigma_{12} + B\sigma_{22})\cup_2]$$

$$E\left(e^{-k\beta}\right) = e^{-k\beta} - e^{-k\beta}k\left(\frac{-\xi}{\beta} + \frac{\gamma}{\beta^2} - \frac{1}{\delta}\right)\frac{Y_{22}}{D} + \left(\frac{\xi-1}{\gamma} + \frac{1}{\beta}\right)\frac{Y_{23}}{D} - \frac{k}{2}\frac{Y_{22}}{D} - \frac{ke^{-k\beta}}{2}(A\sigma_{12} + B\sigma_{22})$$

$$E\left(e^{-k\beta}\right) = e^{-k\beta} + \varphi_4$$

$$\hat{\beta}_{ABL} = -\frac{1}{k}log\left[E_\mu\left(e^{-k\beta}\right)\right]$$

$$\hat{\beta}_{ABL} = \beta - \frac{1}{k}log\varphi_4$$

$$\hat{\beta}_{ABL} = \beta + k'log\varphi_4 \quad ; \qquad \text{at } \left(\hat{\alpha}_{ML,}\hat{\beta}_{ML,}\hat{\gamma}_{ML}\right) \qquad (5.4)$$

Where

$$\varphi_4 = \Bigg[1 - k\left(\frac{-\xi}{\beta} + \frac{\gamma}{\beta^2} - \frac{1}{\delta}\right)\frac{Y_{22}}{D} - \left(\frac{\xi-1}{\gamma} + \frac{1}{\beta}\right)\frac{Y_{23}}{D} - \frac{k}{2}\frac{Y_{22}}{D}\left(\frac{1}{D}Y_{11}\left(\frac{2n}{\alpha^3} - (\gamma+1)\omega_{133}\right) - 2\frac{Y_{12}}{D}(\gamma+1)\omega_{123} - \frac{2Y_{13}}{D}\beta\omega_{122} - \frac{2Y_{22}}{D}(\gamma+1)\omega_{113}\right)\frac{\sigma_{12}}{2} + \frac{\sigma_{22}}{2}\left(-\frac{Y_{11}}{D}(\gamma+1)\omega_{123} - 4\frac{Y_{12}}{D}(\gamma+1)\omega_{113} + 2\frac{Y_{13}}{D}\omega_{112} + \frac{2Y_{23}}{D}\left(\frac{-n}{\beta^2} + \delta_{12}\right) + \frac{Y_{22}}{D}\left(\frac{2n\gamma}{\beta^3} - 2(\gamma+1)\delta_{13}\right)\right)\Bigg] \qquad (5.5)$$

3. Approximate Bayes Estimate of γ

$$\cup(\alpha,\beta,\gamma) = \cup = e^{-k\gamma}$$

$$\cup_1 = \cup_{12} = \cup_{13} = \cup_{11} = 0$$

$$\cup_2 = \cup_{21} = \cup_{22} = \cup_{23} = 0$$

$$\cup_3 = \frac{\partial}{\partial\gamma}\left(e^{-k\gamma}\right) = -ke^{-k\gamma}; \quad \cup_{33} = \frac{\partial}{\partial\gamma}\left(-e^{-k\gamma}\right) = k^2e^{-k\gamma}, \quad \cup_{31} = \cup_{32} = 0$$

$$\varphi_1 = \cup_1 a_1 + \cup_2 a_2 + \cup_3 a_3 + a_4 + a_5$$

$$= -ke^{-k\gamma}\left(\left(\frac{\xi-1}{\gamma} - \frac{1}{\beta} - \frac{k^2}{2}\right)\frac{Y_{33}}{D}\right)$$

$$\varphi_2 = \frac{1}{2}\left[(A\sigma_{11} + B\sigma_{21} + P\sigma_{31})\cup_1 + (A\sigma_{12} + B\sigma_{22} + P\sigma_{32})\cup_2 + (A\sigma_{13} + B\sigma_{23} + P\sigma_{33})\cup_3\right]$$

$$= -ke^{-k\gamma}\left(\frac{A\sigma_{13} + B\sigma_{23} + P\sigma_{33}}{2}\right)$$

$$E\left(u|\underline{x}\right) = e^{-k\gamma} - ke^{-kr}\left(\frac{\xi-1}{\gamma} + \frac{\gamma}{\beta} - \frac{k^2}{2}\right)\frac{Y_{33}}{D} - ke^{-kr}\left(\frac{A\sigma_{13} + \sigma_{23}B + P\sigma_{33}}{2}\right)$$

$$E\left(u|\underline{x}\right) = e^{-k\gamma}\varphi_5$$

$$\hat{\gamma}_{ABL} = -\frac{1}{k}log\left[e^{-k\gamma}\varphi_5\right]$$

$$\hat{\gamma}_{ABL} = \gamma - \frac{1}{k}log\varphi_5,$$

$$\hat{\gamma}_{ABL} = \gamma + k'log\varphi_5; \qquad \text{at } \left(\hat{\alpha}_{ML,}\hat{\beta}_{ML,}\hat{\gamma}_{ML}\right) \qquad (5.6)$$

Where;

$$\varphi_5 = 1 - k\left(\frac{\xi - 1}{\gamma} + \frac{1}{\beta} - \frac{k^2}{2}\right)\frac{Y_{33}}{D} + \left(\frac{A\sigma_{13} + B\sigma_{23} + P\sigma_{33}}{2}\right)$$

Simulation and Numerical Comparison

The simulations and numerical calculations are done by using R Language programming and results are presented in form of tables in table (1).

1. The Random variable of Generalized Compound Rayleigh Distribution is generated by R-Language programming by taking the values of the parameters α, β, γ , taken as $\alpha = 1$, $\beta = 0.5$ and $\gamma = 0.8$ in the equations[(4.2)-(4.4)] and equation(1.1).

2. Taking the different sizes of samples n=10(10)80 with complete sample, MLE's, the Approximate Bayes estimators ,and their respective MSE's (in parenthesis) are obtained by repeating the steps 500 times, are presented in the table (1), for t=0.5, k= -10, R(t)=0.42, H(t)=0.625 and parameters of prior distribution a =2 and b =3.

3. Table (1) presents the MSE of α and β and Approximate Bayes estimators of α and β (for α, β and γ unknown) under LLF. The MSE's in all above cases are presented in parenthesis. Here $\hat{\alpha}_{ABL}$ *and* $\hat{\beta}_{ABL}$ under LLF has the lowest MSE's which shows its domination amongst other estimators.

4. Table (1) presents the MLE of parameter γ of (for known α, β) and Approximate Bayes estimator of γ under Linex Loss function(LLF) (for α, β and γ unknown). The MSE's in all above cases are presented in parenthesis. The estimators have minimum MSE's for small sample sizes, as the sample sizes increase, the MSE's increased. Among all the four estimators $\hat{\beta}_{ABL}$ under LLF has the lowest MSE.

Table (1)

Mean and MSE'S of α, β, γ

$(\alpha = 1 , \beta = 0.5 \text{ and } \gamma = 0.8)$

n	$\hat{\alpha}_{ML}$	$\hat{\alpha}_{ABL}$	$\hat{\beta}_{ML}$	$\hat{\beta}_{ABL}$	$\hat{\gamma}_{ML}$	$\hat{\gamma}_{BL}$	$\hat{\gamma}_{ABL}$
10	0.6845795	0.6999856	0.6584213	0.6125874	0.5900124	0.5999879	0.7995885
	[0.886547]	**[0.895467]**	**[0.052146]**	**[0.056125]**	**[0.0256142]**	**[.0017239]**	**[0.003071]**
20	0.6998754	0.7025461	0.6231444	0.5984521	0.6922601	0.713952	0. 800211
	[0.895647]	**[0.902354]**	**[0.852365]**	**[0.004411]**	**[0.026537]**	**[0.016369]**	**[4.081x10^{-2}]**
30	0.7451688	0.7854789	0.5825888	0.5745896	0.6865478	0. 740252	0. 793433
	[0.057125]	**[0.057812]**	**[0.658458]**	**[0.088541]**	**[0.096548]**	**[0.02589461]**	**[0.4317145]**
40	0.79865421	0.8012125	0.5482658	0.5801811	0.86975682	0. 768581	0. 8937546
	[0.457812]	**[0.045812]**	**[0.002354]**	**[0.005718]**	**[0.003265]**	**[.78954623]**	**[0.8547969]**
50	0.8956874	0.8999574	0.5996584	0.5592358	0.8490011	0.8757613	0.8592358
	[0.004578]	**[0.045821]**	**[0.004577]**	**[0.036142]**	**[0.004265]**	**[0.001624]**	**[0.015437]**
60	0.91478523	0.9141255	0.6990011	0.6792358	0.9490011	0.9757613	0.9792358
	[0.004325]	**[0.004789]**	**[0.054663]**	**[0.055437]**	**[0.004226]**	**[0.001624]**	**[0.015437]**
70	1.000001	1.0254612	0.6988845	0.6807394	0.9454543	0.9924443	0.9807398
	[0.000125]	**[0.000521]**	**[0.001125]**	**[0.010374]**	**[0.001367]**	**[0.003718]**	**[0.010374]**
80	1.2354782	1.0645871	0.7235814	0.7258725	0.9657432	1.0524443	1.007398
	[0.325874]	**[0.254235]**	**[0.025258]**	**[0.013258]**	**[0.001245]**	**[0.004012]**	**[0.010544]**

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