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THE NEW ASSORTMENTS OF HOMOTOPY

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Abstract : In this paper introduction to the homotopy and homotopy equivalence class function. We have proved some theorems and problems for the both functions. Equivalence relation for homotopy function, Group operation and properties for homotopy equivalence class function are discussed.

***Index Terms* - Continuous maps, Equivalence relation, Convex function, Homotopy, Homotopy equivalences.**

I. INTRODUCTION

A notable use of homotopy is the definition of homotopy in algebraic topology. Simply, concerned of homotopy is, the geometric objects which can be continuously deformed into each other and considered equivalent. The continuous transformation from one topological function to another topological function is known as homotopic. Such a continuous deformation being on one function to another is called a homotopy. It was not until the 1920's that a formal expression of such a concept was given in terms of Reflexivity, Symmetry and Transitivity. If the origin of the homotopy concept for path is to be viewed with in the analysis, it was used as a visual tool to determine whether two paths with identical points lead to the same conclusion of integration. Consider the paths on the topological space and an equivalence relation called path homotopy [4]. We shall in mathematics a way of classifying the geometric areas of homotopy we have to study the different types of paths that can be drawn in the areas. Homotopic is the combination of two paths with common nodes, one of which converges to the endpoint if it is continuously distorted into the other. Homotopy groups are started from the principles of homotopy. They are the natural high-dimensional analogies of the basic group. These high-dimensional analogies of homotopy groups have some formal similarities with homology groups [1]. The basic concepts with theorem, problems and examples for the homotopy function and homotopy equivalence function are discussed in this paper.

II. HOMOTOPIC FUNCTIONS

Definition: 2.1. Let A, B be topological spaces and H, H′are continuous maps and there is a continuous map $H, H' : A \to B$. A homotopy from H to H′ is continuous function $M : A \times [0,1] \to B$; $M(\alpha, t)$. Satisfying, $M(\alpha, 0) = H(\alpha)$ and $M(\alpha, 1) = H'(\alpha)$ for all $\alpha \in A$. If such a homotopy exists, then H is homotopic to H′ and its notation by $H \simeq H'$. See Figure 1.

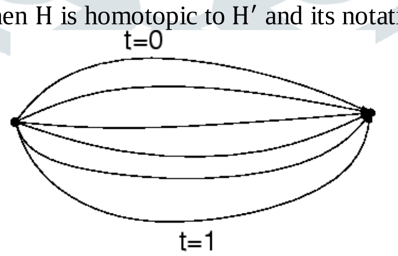

Fig.1

Example: Let H and H′ be any two continuous maps A into real functions. (i.e.)., $H, H' : \mathbb{R} \to \mathbb{R}$ then $H \simeq H'$. Define the function $M : \mathbb{R} \times I \to \mathbb{R}$ by $M(\alpha, t) = (1-t)H(\alpha) + t H'(\alpha)$. Briefly, F is continuous being a composite of continuous function. $M(\alpha, 0) = (1-0)H(\alpha) + 0 H'(\alpha) = H(\alpha)$ and $M(\alpha, 1) = (1-1)H(\alpha) + 1 H'(\alpha) = H'(\alpha)$. Thus M is homotopy between H and H′.

Remark: 2.2. Let H is path in A. If $H : [0,1] \to A$ is continuous map, such that $H(0) = \alpha_0$ and $H(1) = \alpha_1$ then we say that H is a path in A from α_0 to α_1. Here α_0 is the primary point and α_1 is the end point.

Definition: 2.3. The topological spaces H and H′ are continuous and constant map between H is homotopic to H′. Then we say that H is null homotopic.

The space H has the property that, the map is identity then H is said to be contractible. (i.e.)., $i_A : A \to A$ is null homotopic.

Definition: 2.4. If H and H′ be any two continuous maps of the space A into the space B. Two paths H and H′, mapping the interval $I = [0,1]$ into A are said to be path homotopy. If they have the same primary point α_0 and the same end point α_1 and if there is a continuous map $M : I \times I \to A$. Such that $M(u, 0) = H(u)$ and $M(u, 1) = H'(u)$

$M(0, t) = \alpha_0$ and $M(1, t) = \alpha_1$ for each $t, u \in I$. Then M is a path homotopy between H and H′. If H is path homotopic to H′ then the notation $H \simeq_p H'$. See Figure 2.

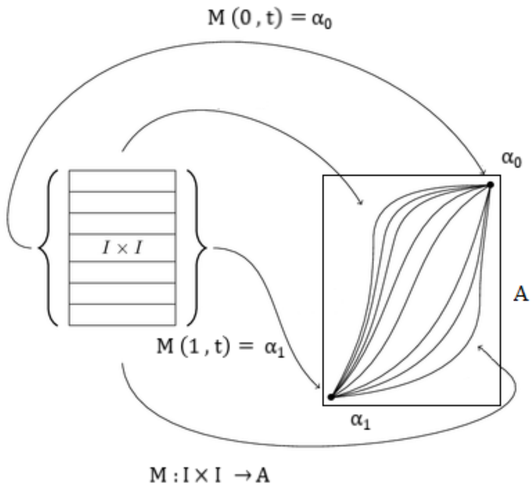

Fig.2

Definition: 2.5. Let H and H′ are any two continuous maps in A. If H is a path in A from α_0 to α_1. If H′ is a path in A from α_1 to α_2. (i.e.)., $H : \alpha_0 \to \alpha_1$ and $H' : \alpha_1 \to \alpha_2$

Now, define the new path product; (i.e.)., $(H * H') : \alpha_0 \to \alpha_2$. The function $(H * H')$ is well defined and continuous by the pasting lemma. The product $(H * H')$ to be the path P given by the equations, $P(u) = \begin{cases} H(2u) & for\ u \epsilon [0, \frac{1}{2}] \\ H'(2u-1) & for\ u \epsilon [\frac{1}{2}, 1] \end{cases}$

Definition: 2.6. The importance of convex sets in terms of homotopy its name is called straight line homotopy or line homotopy. If two paths H and H′ are continuous maps between the points α_0 to α_1 in the convex set C. (i.e.)., H, $H' : \alpha_0 \to \alpha_1$ in C.

In general homotopy between any two paths, $M(u, t) = (1-t)H(u) + tH'(u)$ is a homotopy between them and moves the point H(u) and H′(u) along the straight line segment joining them. It's called a straight line homotopy. See Figure 3.

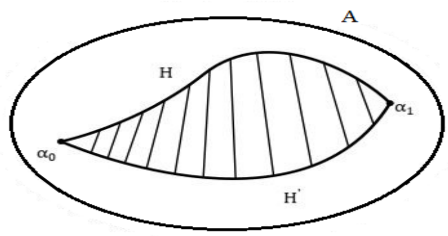

Fig.3

Theorem: 2.7. Let A be the any topological space and let B be a convex subset of $\mathbb{R}^n$, with the subspace topology. Then any two continuous maps H, $H' : A \to B$ are homotopic.

Proof: Let A be any topological space. Let B be a convex subset of $\mathbb{R}^n$. If H and H′ are continuous maps of the space A into the space B. We say that H is homotopic to H′. If there is a continuous map H.

Define the function, $M : \mathbb{R} \times [0, 1] \to \mathbb{R}$

By $M(\alpha, t) = (1-t)H(\alpha) + tH'(\alpha) \epsilon B \subset \mathbb{R}^n$, for every $t \epsilon [0, 1]$

Here M is continuous also $M(\alpha, 0) = H(\alpha)$ and $M(\alpha, 1) = H'(\alpha)$

Then the function M is homotopy between H and H′.

Theorem: 2.8. Homotopy is an equivalence relation on set of continuous map $A \to B$

Proof:

1. **Reflexivity ($H \simeq H'$)**

It is trivial that, $H \simeq H$. The map $M : A \times I \to A$

Then $M(\alpha, t) \Longrightarrow M(\alpha, 0) = H(\alpha)$

$M(\alpha, 1) = H(\alpha)$ for all $\alpha \epsilon A$ is a homotopy from H to H

2. **Symmetry ($H \simeq H' \Longrightarrow H' \simeq H$)**

The map $M : A \times I \to A$ be a homotopy between H and H′. Then the map $N : A \times I \to A$

Then $N(\alpha, t) \Longrightarrow M(\alpha, 1-t)$

$N(\alpha, 0) \Longrightarrow M(\alpha, 1) = H'(\alpha)$

$N(\alpha, 1) \Longrightarrow M(\alpha, 0) = H(\alpha)$ for $\alpha \epsilon A$ is a homotopy from H′ to H

3. **Transitivity ($H \simeq H'$ and $H' \simeq H'' \Longrightarrow H \simeq H''$)**

Let map $M : A \times I \to A$ be a homotopy between H and H′ and $N : A \times I \to A$ is a homotopy between H′ to H″.

Then the map, $P : A \times I \to B$

$$P(\alpha, t) = \begin{cases} M(2t, \alpha) & for\ t \epsilon [0, \frac{1}{2}] \\ N(2t-1, \alpha) & for\ t \epsilon [\frac{1}{2}, 1] \end{cases}$$

The map P is well defined and continuous map. Thus P is a homotopy between $H \simeq H''$

Lemma: 2.9. Homotopy in Compositions

Proof: Let $H, H_1 : \alpha \to \beta$ and $H', H_1' : \beta \to \gamma$

Then $(H \simeq H_1), (H' \simeq H_1') \Longrightarrow (H' \circ H) \simeq (H_1' \circ H_1)$

$$h : H \simeq H_1$$
$$h_1 : H' \simeq H_1'$$

New homotopy $P : (H' \circ H) \simeq (H_1' \circ H_1)$ and $P(\alpha, t) = h_1(h(\alpha, t), t)$

Lemma: 2.10. Product operation - Distributive law in homotopy with composition

Proof: Consider the function h, h_1, h_2 with $*$ operation $(h \circ (h_1 * h_2)) = (h \circ h_1) * (h \circ h_2)$

Define the function,

$$(h \circ (h_1 * h_2))(t) = \begin{cases} h(h_1(2t)) & for\ t \in [0, \frac{1}{2}] \\ h(h_2(2t-1)) & for\ t \in [\frac{1}{2}, 1] \end{cases}$$

Hence, $(h \circ (h_1 * h_2))(t) = \big((h \circ h_1) * (h \circ h_2)\big)(t)$

Theorem: 2.11. Show that $H_1, H_2 : A \to B$ and $H_1', H_2' : B \to \Gamma$ be continuous and homotopic.Then $(H_1' \circ H_1)$, $(H_2' \circ H_2) : A \to \Gamma$ be continuous and homotopic.

Proof: Given that the functions,

$H_1, H_2 : A \to B$ be continuous and let $M : A \times I \to B$ homotopy between H_1 and H_2.

Similarly, $H_1', H_2' : B \to \Gamma$ be continuous and let $M : B \times I \to \Gamma$ homotopy between H_1' and H_2'.

Now we define the map, $P : A \times I \to \Gamma$ be continuous by,

$P(\alpha, t) = N(\alpha, t) = N(M(\alpha, t), t)$, we put $t = 0$ and $t = 1$ and we get

$$P(\alpha, 0) = N(M(\alpha, 0), 0) = N(H(\alpha), 0) = H_1'(H_1(\alpha)) \quad (2.1)$$

$$P(\alpha, 1) = N(M(\alpha, 1), 1) = N(H'(\alpha), 1) = H_2'(H_2(\alpha)) \quad (2.2)$$

From (2.1) and (2.2) $\Longrightarrow (H_1' \circ H_1), (H_2' \circ H_2) : A \to \Gamma$ be continuous and $P : A \times I \to \Gamma$ be homotopy between them.

Theorem: 2.12. Let the spaces $[A, B]$ be the set of homotopy classes of maps A into B. Then show that for any, the set $[A, I]$ has a single element. Where $I = [0, 1]$

Proof: Given that the space $[A, B]$ is homotopy classes of maps A into B which means, If have only two elements in this homotopy classes then for all the maps from A to Y is homotopic to only one if the two possibilities will take.

Theorem: 2.13. Show that the topological space A is contractible iff every map $H : A \to B$ is null homotopic. Similarly, Show that the topological space A is contractible iff every map $H : B \to A$ is null homotopic. Where B is arbitrary.

Proof: In first part, we show that if a space A is contractible, then every map $H : A \to B$ for arbitrary B is null homotopic. Suppose every map $H : A \to B$ for all B is null homotopic. Then the identity map $A \to A$ is null homotopic. So A is contractible. Now, Suppose A is contractible. That is there is a point $\alpha \in A$ and a homotopy $H' : A \times I \to A$, such that H is the identity on A and H' is the constant map with a point α. Also, $H : A \to B$, then $(H \circ H') : A \times I \to B$ is homotopy from H to the map constant with $H(\alpha)$. Thus H is null homotopic.

In second part, we have to prove that if a space A is contractible, then every map $H : B \to A$ for arbitrary B is null homotopic. Suppose that every map $H : B \to A$ for all B is null homotopic. Then the identity map $A \to A$ is null homotopic. So A is contractible. Now, Suppose A is contractible. That is there is a point $\alpha \in A$ and a homotopy $H : B \times I \to B$, such that H' is the identity on B and H is the constant map with a point α. Also, $H' : B \to A$, then $(H' \circ H) : B \times I \to A$ is homotopy from H to the constant map with a value is α. Thus H is null homotopic.

Problem: 2.14. A space A is said to be contractible if the identity map $i_\alpha : A \to A$ is null homotopic.

1. Prove that a contractible space is path connected.

Proof: Let A is a contractible space and let T be the homotopy. By the definition, the identity map on A is homotopic to a constant map at some point $\alpha \in A$ and the homotopy T is between them. Let choose a point $\alpha_0 \in A$ then $T(\alpha_0, t)$ is a path homotopy from α_0 to α. So all the points in A has a path to a general point. Therefore, the space A is path connected.

2. Prove that for any A, the set $[A, B]$ has a single element if B is contractible.

Proof: Given that B is contractible and let T is homotopy. By the definition, the identity map on A is homotopic to a constant map at some point $\alpha \in A$ and the homotopy T is between them. Then for any map $H : A \to B$ such that, $T(\beta, t) = T(H(\alpha), 0) = H(\alpha)$, where $T(\beta, 0)$ is identity map. Finally, $T(H(\alpha), t)$ is homotopy between the map H and the constant map $H' : A \to B$. Hence that the maps in all the points of A is going to a single element α.

Problem: 2.15. Let the topological spaces A and B. Let $[A, B]$ be the set of homotopy classes of maps of A into B.

1. Prove that for any A then the set $[A, I]$, where $I = [0, 1]$ has a single element.

Proof: Let A and B be the topological spaces and let $[A, B]$ be the set of homotopy classes of maps of A into B. Let $H, H' : I \to B$ be continuous maps.

Define the function, $T(\alpha, t) = (1 - t)H(\alpha) + t\,H'(\alpha)$ for $t \in [0, 1]$

Put $t = 0$ Then $T(\alpha, 0) = (1 - 0)H(\alpha) + 0\,H'(\alpha) = H(\alpha) + 0 = H(\alpha)$

Put $t = 1$ Then $T(\alpha, 1) = (1 - 1)H(\alpha) + 1\,H'(\alpha) = 0 + H'(\alpha) = H'(\alpha)$

Hence it's a path homotopy from H to H' and I is a convex subset of $\mathbb{R}$ has a single element.

2. Prove that if A is path connected, the set $[I, B]$ has a single element.

Proof: Given that the topological spaces are A and B. Let $[A, B]$ be the set of homotopy classes of maps of A into B. Let $H : I \to B$ be continuous function. Next we have to prove that, H is homotopic to the constant map at $H(0)$.

Define the function, $\Gamma_{H(0)} : I \to B$ by $\Gamma_{H(0)} = H(0)$

Now fix a path by $(1 - t)\alpha$ from α to 0 and $h : I \times I \to Y$ by $h(\alpha, t) = H((1 - t), \alpha)$ where h is continuous. We put $t = 0$ and $t = 1$ here we get, $h(\alpha, 0) = H((1 - o)\,\alpha) = H(\alpha)$

$h(\alpha, 1) = H((1 - 1)\,\alpha) = H(0)$. Here H is homotopic to the constant map based at some points.

III. HOMOTOPIC EQUIVALECE CLASS FUNCTIONS

Definition: 3.1. Two spaces A and B are of the same homotopy. Let H: A → B be a continuous map then H is called homotopy equivalence. If there exists a continuous map H′: B → A. Such that $(H \circ H') \simeq id_\beta$ or identity : $A \to$ A and $(H' \circ H) \simeq id_\alpha$ or identity : $B \to$ B. See Figure 4.

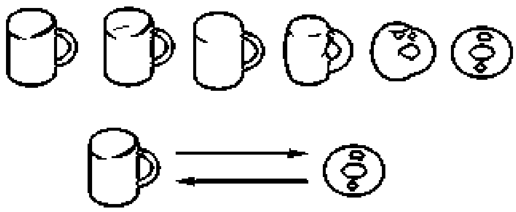

Fig.4

Remark: 3.2. Homotopy equivalent written $A \simeq B$ and Homotopy equivalence written H : A → B

Remark: 3.3. Two homeomorphic topological spaces are of the same homotopy type but the converse is not true [2].

Remark: 3.4. Continuous map H' is called homotopy inverse. (i.e.), H′ : B → A

Definition: 3.5. This path product on equivalence class is very important because this will have lots of properties with equivalence classes. Let the two different paths are [H]and [H′]. Then the product operation on path induces a well-defined operation on path homotopy equivalence classes.

By the equation, $[H] * [H'] = [H * H']$

Let h be a path homotopy between h_1 and h_2 (i.e.)., $h : \alpha_1 \simeq_p \alpha_2$

Let h' be a path homotopy between $h_1{}'$ and $h_2{}'$ (i.e.)., $h' : \beta_1 \simeq_p \beta_2$

Then $P(u,t) = \begin{cases} h(2u,t) & for\ u \epsilon [\,0, \frac{1}{2}] \\ h'(2u-1,t) & for\ u \epsilon [\frac{1}{2},\ 1\,] \end{cases}$

The equivalence class function is well-defined and continuous by pasting lemma.

Proposition: 3.6.

Consider the set of two different equivalence class functions [H], [H′] and $[H_1], [H_1{}']$

Here, $[H] * [H'] = [H * H']$ and $[H_1] * [H_1{}'] = [H_1 * H_1{}']$

Prove that $[H * H']$ and $[H_1 * H_1{}']$ equals or not if $[H] = [H_1]$ and $[H'] = [H_1{}']$

Proof: These two functions are equals to each other it’s just have to be homotopy.

$h : H \simeq_p H_1$ since $[H] = [H_1]$ same equivalence class.

$h_1 : H' \simeq_p H_1{}'$ since $[H'] = [H_1{}']$ same equivalence class. $P : (H * H') \simeq_p (H_1 * H_1{}')$

We define the function, $P(u,t) = \begin{cases} h(2u,t) & for\ u \epsilon [\,0, \frac{1}{2}] \\ h'(2u-1,t) & for\ u \epsilon [\frac{1}{2},\ 1\,] \end{cases}$

From the path homotopy $P : (H * H') \simeq_p (H_1 * H_1{}')$

Proposition: 3.7. If the equivalence class H, H_1 and the equivalence class H′, $H_1{}'$ are equal. Show that the equivalence classes $[H * H'] = [H_1 * H_1{}']$

Proof: Given that two different equivalence class objects with ∗ operator,

$$\text{So } [H] * [H'] = [H * H'] \tag{3.1}$$

$$\text{And } [H_1] * [H_1{}'] = [H_1 * H_1{}'] \tag{3.2}$$

This is set of different functions. Where this equivalence classes are equal but the two functions are not equal. And remember that the equation (3.1) and (3.2) equals to be each other and have a homotopic.

If [H], [H′] is same homotopy equivalence class then the path homotopic $l : H \simeq_p H_1$

If $[H_1], [H_1{}']$ is same homotopy equivalence class then the path homotopic $m : H_1 \simeq_p H_1{}'$

Finally the path homotopic, $P : (H * H') \simeq_p (H_1 * H_1{}')$

Define the transitive homotopy,

$$P(u,t) = \begin{cases} l(2u,t) & for\ u \epsilon [\,0, \frac{1}{2}] \\ m(2u-1,t) & for\ u \epsilon [\frac{1}{2},\ 1\,] \end{cases}$$

It is satisfies all of the properties of path homotopy. So we get the new path homotopy, $P : H * H' \simeq_p H_1 * H_1{}'$.

Therefore, $[H * H'] = [H_1 * H_1{}']$ two homotopy classes are equals to each other.

Proposition: 3.8. Show that the Group operations in equivalence class

1. Associtivity

Consider the paths h_1, h_2, h_3 with continuous and $([h_1] * [h_2]) * [h_3] = [h_1] * ([h_2] * [h_3])$, here we have to prove that the left hand-side path is homotopic to right hand side path.

$$(([h_1] * [h_2]) * [h_3])(t) = \begin{cases} ([h_1] * [h_2])(2t) & for \quad t \epsilon [\,0, \frac{1}{2}\,] \\ [h_3](2t-1) & for \quad t \epsilon [\frac{1}{2}, 1\,] \end{cases}$$

$$= \begin{cases} [h_1]\,(4t) & for \quad t\,\epsilon\,[\,0,\frac{1}{4}\,] \\ [h_2]\,(4t-1) & for \quad t\,\epsilon\,[\frac{1}{4},\,\frac{1}{2}] \\ [h_3]\,(2t-1) & for \quad t\,\epsilon\,[\frac{1}{2},1\,] \end{cases}$$

$$(\,[h_1]\,*(\,[h_2]*[h_3]\,)\,)\,(t) = \begin{cases} [h_1]\,(2t) & for \quad t\,\epsilon\,[\,0,\frac{1}{2}\,] \\ [h_2]*[h_3]\,(2t-1) & for \quad t\,\epsilon\,[\frac{1}{2},1\,] \end{cases}$$

$$= \begin{cases} [h_1]\,(2t) & for \quad t\,\epsilon\,[\,0,\frac{1}{2}\,] \\ [h_2]\,(4t-2) & for \quad t\,\epsilon\,[\frac{1}{2},\,\frac{3}{4}] \\ [h_3]\,(4t-3) & for \quad t\,\epsilon\,[\frac{3}{4},1\,] \end{cases}$$

So we get these two different functions and we need to create a homotopy between them. And it is defined by the function $P : I \times I \to A$

$$P(u,t) = \begin{cases} [h_1]\,((4-2u)\,t) & for \quad t\,\epsilon\,[\,0,\frac{1}{4-2u}\,] \\ [h_2]\,(4t-(1+u)) & for \quad t\,\epsilon\,[\frac{1}{4-2u},\,\frac{2+u}{4}] \\ [h_3]\,((2+2u)t-(1+2u)) & for \quad t\,\,\epsilon\,[\frac{2+u}{4},1\,] \end{cases}$$

Hence, this is homotopy between the paths $(\,[h_1]\,*[h_2]\,)\,*\,[h_3]$ and $[h_1]\,*(\,[h_2]*[h_3]\,)$ also the function is well-defined and continuous by pasting lemma.

2. Identity

Let the map $e_{\alpha_0} : I \to A$ and every single value $v \to \alpha_0$. Here I is interval and A is topological space.

Then : $\alpha_0 \to \alpha_1$, the equivalence class $[e_{\alpha_0}] * [H] = [H]$

$H : \alpha_1 \to \alpha_0$, two sided identity equivalence class $[H] * [e_{\alpha_1}] = [H]$

Now using the identity map $i : I \to I$, The identity path $i : 0 \to 1$ and the constant function $e_0 : 0 \to 0$

Therefore, $(e_0 * i) : 0 \to 1$, I is convex .

That means, any two paths between the same two points are homotopy. $(e_0 * i) \simeq_p i$ (3.3)

By using lemma (2.9). Present this homotopy preserve to prove composition into path homotopy.

From (3.3) Taking both sides on h compositions, $h(e_0 * i) \simeq_p h(i)$

$$(h \circ e_0) * (h \circ i) \simeq_p (h \circ i)$$

$$(e_0 * h) \simeq_p h$$

$$[e_{\alpha_0}] * [h] = [h]$$

Similarly 2nd part of proof.

3. Inverse

Let the inverse map $\bar{\imath} : 1 \to 0$, (i.e.)., $i : 0 \to 1$

Therefore, $(i\,*\,\bar{\imath}\,) : 0 \to 0$, since $e_0 : 0 \to 0$

$(i\,*\,\bar{\imath}\,) \simeq_p e_0$ (3.4)

From (3.4) Taking both sides on h compositions, $h(i\,*\,\bar{\imath}\,) \simeq_p h(e_0\,)$

$(h \circ i)\,*\,(h \circ \bar{\imath}) \simeq_p (\mathrm{h} \circ e_0\,)$

$(h * \bar{h}\,) \simeq_p e_{\alpha_0}$

Therefore, the equivalence class products; $[h]\,*\,[\bar{h}] = e_{\alpha_0}$

Similarly 2nd part of proof.

Remark: 3.9. $\bar{\imath}(s) = i(1-s) = (1-s)$

Here, $(h\,\circ\,\bar{\imath})(s) = f(\bar{\imath}(s)) = f(1-s) = \bar{h}(s)$

IV. CONCLUSION

In this paper, the basic definitions required to solve the problems and theorems are mentioned. It is also discussed with properties for the homotopy and homotopy equivalence functions. Following this, the homotopy is represented by some problems and theorems using the basic definitions.

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