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THE HAHN BANACH THEOREM IN FUNCTIONAL ANALYSIS

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ABSTRACT

The present article explains proof of The Hahn-Banach theorem which is one of the most important theorems in functional analysis . In Hahn-Banach theorem we have shown that all of the bounded linear functionals on a Hilbert space are just the scalar products, you might begin to wonder whether Banach spaces which are not Hilbert spaces have any non,zero bounded linear functionals at all. (In fact, one might be tempted to reason as follows: if the only nontrivial bounded linear functionals on a Banach space that has a scalar product are precisely the scalar products, then Banach spaces which do not have scalar products should have none.) we consider the simplest case possible.

Let X be a Banach space, and let $x_0 \neq 0$ be a fixed element of X. The set of all elements of the form αx_0 forms a subspace X_0 of X. One candidate is

$$F(\alpha x_0) = \alpha.$$

Clearly this is a linear functional. It is also bounded, sinc-e

$$|F(\alpha x_0)| = |\alpha| = \frac{||\alpha x_0||}{||x_0||}.$$

So there are bounded linear functionals on such $\text{subspac}_1\text{es}$. If we could only extend them to the whole of X we would have what we want. However, difficulties immediately present themselves. Besides the fact that it is not obvious how to extend a bounded linear functional to larger subspaces, will the norm of the functional be increased and after applying infinite number of steps to complete the procedure . The answers to these questions are given by the celebrated Hahn-Banach theorem

Key words: bounded , functional, sublinear , vector space , ambitious , norm.

INTRODUCTION

Let V be a vector space. A functional $p(x)$ on V is called *sublinear* if

$$p(x+y) \leq p(x) + p(y)\,, x, y \in V,$$

(1.1)

$$p(\alpha x) = \alpha p(x)\,, x \in V, \alpha > 0.$$

Note that the norm in a normed vector space is a sublinear functional.

Theorem (*The Hahn-Banach Theorem*) *Let V be a vector space, and let $p(x)$ be a sublinear functional on V. Let M be a subspace of V, and let $f(x)$ be a tinear functíonal on M satisfying*

(1.2) $$f(x) \leq p(x)\,, x \in M.$$

Then there is a linear functional $F(x)$ on the whole of V such that

(1.3) $$F(x) = f(x)\,, x \in M,$$

(1.4) $$F(x) \leq p(x)\,, x \in V.$$

Before attempting to prove the Hahn-Banach theorem, we shall show how it applies to our case. We have

Theorem *Let M be a subspace of a normed vector space X, and suppose that $f(x)$ is a bounded linear functional on M. Set*

$$\|f\| = \sup \frac{|f(x)|}{\|x\|}.$$

Then there is a bounded linear functional $F(x)$ on the whole of X such that

(1.5) $$F(x) = f(x)\,, x \in M,$$

(1.6) $$\|F\| = \|f\|.$$

Proof. Set $$p(x) = \|f\| \cdot \|x\|, x \in X.$$

Then $p(x)$ is a sublinear functional and

$$f(x) \leq p(x)\,, x \in M.$$

Then by the Hahn-Banach theorem there is afunctional $F(x)$ defined on the whole of X such that (1.5) holds and

Since

$$F(x) \leq p(x) = \|f\| \cdot \|x\|, x \in X.$$

we have

$$-F(x) = F(-x) \leq \|f\|\,\| - x\|, x \in X,$$

Thus,

$$|F(x)| \leq \|f\|\,\|x\|, x \in X.$$

$$\|F\| \leq \|f\|.$$

Since F is an extension of f, we must have

$$\|f\| \leq \|F\|.$$

Hence, (1.6) holds, and the proof is complete.

Since we have shown that every normed vector space having nonzero elements has a subspace having a nonzero bounded linear functional, it follows that every $Reject$normed vector space having nonzero elements has nonzero bounded linear functionals.

Now we tackle the Hahn-Banach theorem. We first note that it says nothing if $M = V$. So we assume that there is an element x_1 of V which is not in M. Let M_1 be the set of elements of V of the form

(1.7) $$\alpha x_1 + x, \alpha \in \mathbb{R}, x \in M.$$

Then one checks easily that M_1 is a subspace of V and that the representation (1.7) is unique. To feel our way, let us consider the less ambitious task of extending f to M_1 so as to preserve

(1.2). If such an extension F exists on M_1, it must satisfy

$$F(\alpha x_1 + x) = \alpha F(x_1) + F(x) = \alpha F(x_1) + f(x) .$$

Therefore, F is completely determined by the choice of $F(x_1)$. Moreover, we must have

(1.8) $$\alpha F(x_1) + f(x) \le p(\alpha x_1 + x)$$

for all scalars α and $x \in M$. If $\alpha > 0$, this -means

$F(x_1) \le \frac{1}{\alpha}[p(\alpha x_1 + x) - f(x)] = p\left(x_1 + \frac{x}{\alpha}\right) - f\left(\frac{x}{\alpha}\right) = p(x_1 + z) - f(z)$, where $z = x/\alpha$. If $\alpha < 0$, we have

$$F(x_1) \ge \frac{1}{\alpha}[p(\alpha x_1 + x) - f(x)] = f(y) - p(-x_1 + y) ,$$

where $y = -x/\alpha$. Thus we need

(1.9) $f(y) - p(y - x_1) \le F(x_1) \le p(x_1 + z) - f(z)$ for all $y, z \in M$.

Conversely, if we can pick $F(x_1)$ to satisfy (1.9), then it will satisfy (1.8), and F will satisfy (1.4) on M_1. For if $F(x_1)$ satisfies (1.9), then for $\alpha > 0$, we have

$\alpha F, (x_1) + f(x) = \alpha\left[F(x_1) + f\left(\frac{x}{\alpha}\right)\right] \le \alpha p\left(x_1 + \frac{x}{\alpha}\right) = p(\alpha x_1 + x)$,

while for $\alpha < 0$ we have

$\alpha F(x_1) + f(x) = -\alpha\left[-F(x_1) + f\left(-\frac{x}{\alpha}\right)\right] \le -\alpha p\left(-\frac{x}{\alpha} - x_1\right) = p(\alpha x_1 + x)$. So we have now reduced the problem to finding avalue of $F(x_1)$ to satisfy (1.9). In order for such avalue to exist, we must have

(1.10) $$f(y) - p(y - x_1) \le p(x_1 + z)_1 - f(z)$$

for all $y, z \in M$. In other words we need

$$f(y+z) \leq p(x_1+z) + p(y-x_1)\,.$$

This is indeed true by (1.2) and property (1.0) of asublinear functional. Hence, (1.10) holds. If we fix y and let z run through all elements of M, we have

$$f(y) - p(y-x_1) \leq \inf\,[p(x_1+z) - f(z)] \equiv C.$$

Since this is true for any $y \in M$, we have

$$c \equiv \sup\,[f(y) - p(y-x_1)] \leq C.$$

We now merely pick $F(x_1)$ to satisfy

$$c \leq F(x_1) \leq C.$$

Note that the extension F is unique only when $c = C$.

Thus we have been able to extend f from M to M_1 in the desired way.

If $M_1 = V$, we are finished. Otherwise there is an element x_2 of V not in M. Let M_2 be the space "spanned" by x_2 and M_1. By repeating the process we can extend f to M_2 in the desired way. If $M_2 \neq V$, we keep a stiff upper lip and continue. We get a sequence M_k of subspaces each containing the preceding and such that f can be extended from one to the next. If, finally, we reach a k such that $M_k = V$, we are finished. Even if

(1.11) $$V = \cup_{k=1}^{\infty} M_k,$$

then we are through because each $x \in V$ is in some M_k, and we can define F by induction. But what if (1.11) does not hold? We can cQmplete the proof easily when V is a Hilbert space and $p(x) = \gamma\|x\|$ for some positive constant γ. For then one can extend f to the closure $\overline{M}$ of M by continuity. By this we mean that if $\{x_n\}$ is a sequence of elements in $M_\backslash$which converges to $x \in V$, then $\{f(x_n)\}$ is a Cauchy sequence of real numbers and hence has a limit. Wc then define $F(x) = \lim f\,(x_n)$. The limit is 'independent of the sequence chosen.

One checks easily that $F(x)$ is a bounded linear functional on the set $\overline{M}$ and coincides with $f(x)$ on ΛI, Since $\overline{M}$ is a Hilbert space, there is an element $y \in \overline{M}$ such that $F(x) = (x, y)$ for all $x \in \overline{M}$ (Theorem 1.). Moreover $\|y\| = \|f\| \leq \gamma$. But $F(x)$ can be defined as (x, y) on the whole of V, and its norm will not be increased.

when the space V is not a Hilbert space and (1.11) does not hold.in this case This is not a trivial situation. In this case we need a statement known as Zorn's lemma concerning maximal elements of chains in partially ordered sets. This lemma is equivalent to the axiom of choice, the validity of which we cannot discuss here. We shall present Zorn's lemma. There we shall complete the proof of the Hahn-Banach theorem for those spaces that require Zorn's lemma in the proof.

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