JETIR.ORG ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue
JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)
An International Scholarly Open Access, Peer-reviewed, Refereed Journal

Riesz representation theorem in Hilbert space

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ABSTRACT

This paper related to principal of functional analysis that explains the original topics or materials that can be developed by mean of techniques existing within the original framework .In particular we discuss normed vector spaces and linear operators specially results related to Hilbert space i.e. The Riesz representation theorem & many other examples of bounded linear functionals

Key words: scalar product, vector space, functional, bounded, closed,orthogonal, Schwarz's inequality, vanish.

INTRODUCTION

Let H be a Hilbert space and let (x, y) denote its scalar product. If we fix y, then the expression (x, y) assigns to each $x \in H$ a number. An assignment F of a number to each element x of a vector space is called a *functional* and denoted by $F(x)$. The scalar product is not the first functional we have encountered. In any normed vector space, the norm is also a functional.

The functional $F(x) = (x, y)$ has some very interesting and surprising features. For instance it satisfies

$$(1.1) \qquad F(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 F(x_1) + \alpha_2 F(x_2)$$

for α_1, α_2 scalars. A functional satisfying (1.1) is called *linear*. Another property is

$$(1.2) \qquad |F(x)| \leq M\|x\|,$$

which follows immediately from Schwarz's inequality. A functional satisfying (1.2) is called *bounded*. Thus for y fixed, $F(x) = (x, y)$ is a bounded linear functional in the Hilbert space H.

Now we can find many other examples of bounded linear functionals Well. In fact we have the following

Theorem 1.1. (*Riesz representation theorem*) *For every bounded linear functíonal* F *on a Hílbert space* H *there is a uruque element* $y \in H$ *such that*

(1.3) $F(x) = (x, y)$ for all $x \in H$.

Moreover,

(1.4) $\|y\| = \sup \frac{|F(x)|}{\|x\|}$.

In order to get an idea how to go about proving it, let us examine (1.3) a bit more closely. If F assigns to each element x the value zero, then we can take $y = 0$, and the theorem is trivial. Otherwise the y we are searching for cannot vanish. However, it must be "orthogonal" to every x for which $F(x) = 0$; i.e., we must have $(x, y) = 0$ for all such x. Let N denote the set of those x satisfying $F(x) = 0$. Suppose we can find a $y \neq 0$ which is orthogonal to each $x \in N$. Then I claim that the theorem is proved. For clearly y is not in N (otherwise we would have $\|y\|^2 = (y, y) = 0$), and hence $F(y) \neq 0$. Moreover, for each $x \in H$, we have

$$F(F(y)x - F(x)y) = F(y)F(x) - F(x)F(y) = 0,$$

showing that $F(y)x - F(x)y$ is in N. Hence

$$(F(y)x - F(x)y, y) = 0,$$

or

$$F(x) = \left(x, \frac{F(y)}{\|y\|^2} y\right).$$

This gives (1.3) if we use $F(y)y/\|y\|^2$ in place of y. ⟨This is to be expected since we made no stipulation on y other than that it be orthogonal to N.) We also note that the uniqueness and (1.4) are trivial. For if y_1 were another element of H satisfying (1.3), we would have

$(x, y - y_1) = 0$ for all $x \in H$.

In particular this holds for $x = y - y_1$, showing that $\|y - y_1\| = 0$. Thus, $y_1 = y$. Now by Schwarz's inequality,

Hence,

$$|F(x)| = |(x, y)| \leq \|x\|\|y\|.$$

$$\|y\| \geq \sup \frac{|F(x)|}{\|x\|}.$$

However, we can obtain equality by taking $x = y$. In fact, $\|y\| = |F(y)|/\|y\|$. This gives (1.4).

In order to do element $y \neq 0$ which is orthogonal to N (i.e., to every element of N) . we examine N a little more closely. What kind of set is it? First of all, we notice that if x_1 and x_2 are elements of N, so is $\alpha_1 x_1 + \alpha_2 x_2$ for any scalars α_1, α_2. For, by the linearity of F,

$$F(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 F(x_1) + \alpha_2 F(x_2) = 0.$$

A subset U of a vector space V is called a *subspace* of V if $\alpha_1 x_1 + \alpha_2 x_2$ is in U whenever x_1, x_2 are in U and α_1, α_2 are scalars. Thus N is a subspace of H. There is another property of N which comes from (1.2) and is not so obvious. This is the fact that it is a closed subspace. A subset U of a normed vector space X is called *closed* if for every sequence $\{x_n\}$ of elements in U having a limit in X, the limit is actually in U. In our particular case, if $\{x_n\}$ is a sequence of elements in N which approaches a limit x in H, then by (1.2)

$|F(x)| = |F(x) - F(x_n)| = |F(x - x_n)| \leq M\|x - x_n\| \to 0$ as $n \to \infty$. Since x does not depend on n, we have $F(x) = 0$. Thus, $x \in N$, showing that N is closed in H.

Thus, we have a closed subspace N of H which is not the whole of H. We

are interested in obtaining an element $y \neq 0$ of H which is orthogonal to N. For the special case of two-dimensional Euclidean space, we recall from our plane geometry that this can be done by drawing a perpendicular. We also recall that the shortest distance from a point (element) to a line (subspace) is along the perpendicular. The same thing is true in Hilbert space. We have

Theorem 1.1. *Let N be a closed subspace of a Hilbert space H. and let x be an element of H which is not in N. Set*

(1.5)

$$d = \inf \|x - z\|.$$

Then there is an element $z \in N$ such that $\|x - z\| = d$.

Proof. By the definition of d, there is a sequence $\{z_n\}$ of elements of N such that $\|x - z_n\| \to d$. We apply the parallelogram law to $x - z_n$ and $x - z_m$. Thus

$$\|(x - z_n) + (x - z_m)\|^2 + \|(x - z_n) - (x - z_m)\|^2 = 2\|x - z_n\|^2 + 2\|x - z_m\|^2,$$

or

(1.6) $4\|x - [(z_n + z_m)/2]\|^2 + \|z_m - z_n\|^2 = 2\|x - z_n\|^2 + 2\|x - z_m\|^2\backslash$

Since N is a subspace, $(z_n + z_m)/2$ is in N. Hence, the left-hand side of (1.6) is not less than

$$4d^2 + \|z_m - z_n\|^2$$

This implies

$\|z_m - z_n\|^2 \leq 2\|x - z_n\|^2 + 2\|x - z_m\|^2 - 4d^2 \to 0$ as $m, n \to \infty$.

Thus, $\{z_n\}$ is a Cauchy sequence in H. Using the fact that a Hilbert space is complete, we let z be the limit of this sequence. But N is closed in H. Hence, $z \in N$, and $d = \lim \| x - z_n\| = \|x - z\|$.

Theorem 1.3. *Let N be a closed subspace of a Hilbert space H. Then for each $x \in H$, there are a $v \in N$ and a w orthogonal to N such that $x = v + w$. This decomposition is unique.*

Proof. If $x \in N$, put $v = x$, $w = 0$. If $x \notin N$, let $z - \in N$ be such that $\|x - z\| = d$, where d is given by (1.5). We set $v = z$, $w = x - z$ and must show that w is orthogonal to N. Let $u \neq 0$ be any element of N and α any scalar. Then

$$d^2 \leq \|w + \alpha u\|^2 = \|w\|^2 + 2\alpha(w,u) + \alpha^2\|u\|^2$$

$$= \|u\|^2 \left[\alpha^2 + 2\alpha\frac{(w,'u)}{||u||^2} + \frac{(w,'u)^2}{\|u\|^4}\right] + d^2 - \frac{(w,'u)^2}{\|u\|^2}$$

$$= \|u\|^2 \left[\alpha + \frac{(w,'u)}{||u||^2}\right]^2 + d^2 - \frac{(w,'u)^2}{\|u\|^2},$$

where we completed the square with respect to α. Take $\alpha = -(w,u)/\|u\|^2$. Thus, $(w,u)^2 \leq 0$, which can only happen if w is orthogonal to u. Since u was any arbitrary element of N, the first statement is proved. If $x = v_1 + w_1$, where $v_1 \in N$ and w_1 is orthogonal to N, then $v - v_1 = w_1 - w$ is both in N and orthogonal to N. In particular, it is orthogonal to itself and thus must vanish. This completes the proof. □

The proof of Theorem 1.1 (the Riesz Representation Theorem) is now complete.

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