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SOME NEW GRAPH OPERATIONS AND THEIR FIRST AND SECOND ZAGREB AND GOURAVA INDICES

[1]Ram Prakash Somani and Varshiki Jethwani

Department of Mathematics, Government College Kota-324001, Rajasthan

(Affiliated to University of Kota, Kota)

Correspondence should be addressed to Varshiki Jethwani; varshikijethwani@gmail.com

Abstract

Chemical graph theory is a fascinating branch of graph theory which has many applications related to chemistry. A topological index is a real number related to a graph, as it's considered a structural invariant. It's found that there is a strong correlation between the properties of chemical compounds and their topological indices. In this paper, we first introduce some new graph operations and compute the values for the first and second Zagreb index.

We then introduce the first and second Gourava indices of a molecular Graph and compute these indices for the aforementioned graph operations. At last, as an example, we will find the Gourava index of binary operations on the Cyclic and the Complete Graph.

Keywords: Chemical Graph theory, Topological indices**,** First and Second Zagreb index, Gourava index, Binary operations, Complete Graph, Cyclic Graph

Introduction and Preliminaries

Let G be a simple and connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set E(G) such that $|E(G)| = m$. If any two vertices $u, v \in V(G)$ are adjacent with an edge, they are denoted by $uv \in E(G)$. Degree of any vertex $v \in V(G)$ is the number of edges that are incident to v and denoted by $deg_G(v)$.

In chemical graph theory, different chemical structures are usually modelled by a molecular graph to understand different properties of the chemical compound theoretically. A graph invariant that correlates the physio-chemical properties of a molecular graph with a number is called a molecular structure index. By use of the adjacency, degree, or distance matrices in graph theory, one can describe the structure of molecules in chemistry using vertex degree based topological indices and distance based topological indices. [3,4,5,7,8].

[1] rpsomani67@gmail.com

The first and second Zagreb indices, first appeared in a topological formula for the total π-energy of conjugated molecules, were introduced by Gutman et al. in [3]. These indices have been used as branching indices. The Zagreb indices have found
applications in QSPR and QSAR studies.

The First and Second Zagreb Indices of Graphs are defined as follows:

$$M_1(G) = \sum_{uv\in E(G)}[d_G(u) + d_G(v)] \quad \text{or} \quad \sum_{u\in V(G)}[d_G^2(u)]$$
$$M_2(G) = \sum_{uv\in E(G)}[d_G(u).d_G(v)]$$

Motivated by the definitions of the Zagreb indices and their wide applications,
V.R Kulli introduced the first and Second Gourava index of a molecular graph in [6] as follows:

$$GO_1(G) = \sum_{uv\in G}\left[(d_G(u) + d_G(v)) + d_G(u).d_G(v)\right],$$
$$GO_2(G) = \sum_{uv\in G}\left[(d_G(u) + d_G(v)).d_G(u).d_G(v)\right]$$

In this section, we introduce definitions and some properties of graphs (resulted by the new binary operations introduced in [5]).

Binary Operations

Binary operations can be broadly classified into 2 types- Unary and Binary

1.) Unary operations create a new graph from the old one, such as addition or deletion of a vertex or an edge, Complement, transpose, power of graph, line graph, dual graph etc.

2.) A binary operation creates a new graph from two initial graphs, such as classic operations (tensor product, cartesian product, strong product, composition, disjunction and symmetric difference).

New Binary Graph Operations

The new binary operations on graphs [10] denoted $\otimes_i$ where $i\in \{0,1,2, ..., 7\}$, defined as follows:

If G_1 and G_2 are two graphs. Then, the vertex sets are defined as follows
$V(G_1\otimes_i G_2) = V(G_1) \times V(G_2)$,

Whereas, the edge sets are defined as follows:

(1) $E(G_1\otimes_0 G_2) = \{(a, b)\ (c, d): [ac \in E(G_1), bd \in E(G_2))]\}$,
(2) $E(G_1\otimes_1 G_2) = \{(a, b)\ (c, d): [ac \in E(G_1), b = d]\}$,
(3) $E(G_1\otimes_2 G_2) = \{(a, b)\ (c, d): [ac \in E(G_1), bd \in E(G_2^c)]\}$,
(4) $E(G_1\otimes_3 G_2) = \{(a, b)\ (c, d): [a=c, bd \in E(G_2)]\}$,
(5) $E(G_1\otimes_4 G_2) = \{(a, b)\ (c, d): [a=c, bd \in E(G_2^c)]\}$,
(6) $E(G_1\otimes_5 G_2) = \{(a, b)\ (c, d): [ac \in E(G_1^c), bd \in E(G_2)]\}$,
(7) $E(G_1\otimes_6 G_2) = \{(a, b)\ (c, d): [ac \in E(G_1^c), b = d]\}$,
(8) $E(G_1\otimes_7 G_2) = \{(a, b)\ (c, d): [ac \in E(G_1^c), bd \in E(G_2^c)]\}$

Lemma 1. Let two graphs $G_1$and G_2 where; $|V(G_1)| = p_1$, $|V(G_2)| = p_2$, $|V(G_1^c)| = p_1$, $|V(G_2^c)| = p_2$, $|E(G_1)| = q_1$ and $|E(G_2)| = q_2$, $|E(G_1^c)| = q_1^c$, $|E(G_2^c)| = q_2^c$, a, c $\in V(G_1)$ and b, d$\in V(G_2)$ Then,

(1) $|E(G_1\otimes_0 G_2)| = 2q_1q_2$,
(2) $|E(G_1\otimes_1 G_2)| = q_1p_2$,
(3) $|E(G_1\otimes_2 G_2)| = 2q_1q_2^c = p_2^2q_1 - p_2q_1 - 2q_1q_2$,
(4) $|E(G_1\otimes_3 G_2)| = q_2p_1$,
(5) $|E(G_1\otimes_4 G_2)| = q_2^c\ p_1 = \frac{1}{2}(p_1p_2^2 - p_1p_2 - 2p_1q_2)$,
(6) $|E(G_1\otimes_5 G_2)| = 2q_1^cq_2 = p_1^2q_2 - p_1q_2 - 2q_1q_2$,

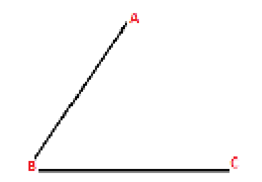
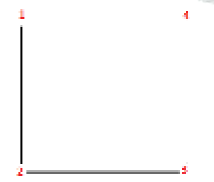
(7) $|E(G_1\otimes_6G_2)| = q_1^c p_2 = \frac{1}{2}(p_2p_1^2 - p_1p_2 - 2p_2q_1)$,

(8) $|E(G_1\otimes_7G_2)| = 2q_1^c q_2^c = \frac{1}{2}[p_1(p_1-1) - 2q_1]\,[p_2(p_2-1) - 2q_2]$

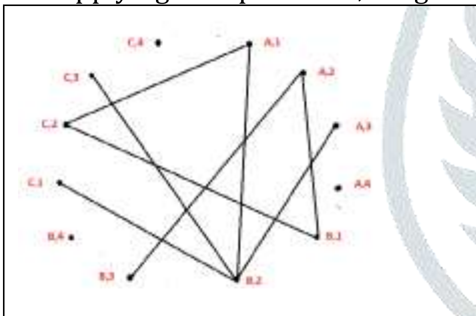
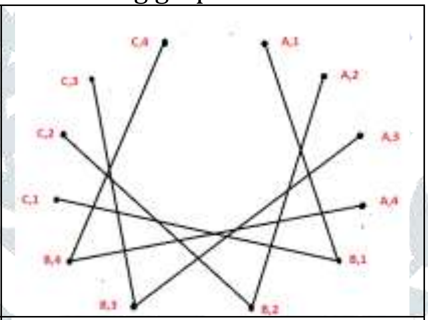
Lemma 2. Let two graphs G_1 and G_2 where; $|V(G_1)| = p_1$, $|V(G_2)| = p_2$, $|V(G_1^c)| = p_1$, $|V(G_2^c)| = p_2$, $|E(G_1)| = q_1$ and $|E(G_2)| = q_2$, $|E(G_1^c)| = q_1^c$, $|E(G_2^c)| = q_2^c$, a, c ∈ $V(G_1)$ and b, d∈ $V(G_2)$ Then,

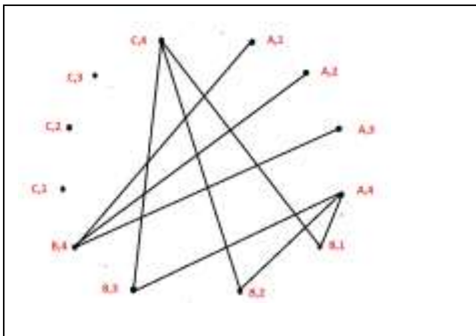
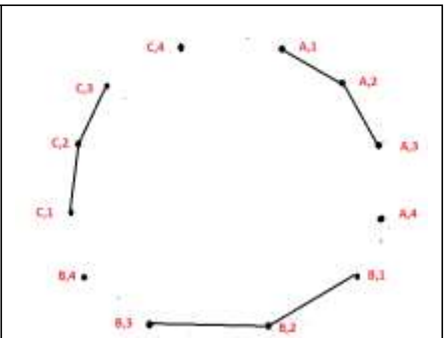
[2](1) $dG_1\otimes_0G_2(u, v) = dG_1(u)dG_2(v)$,
(2) $dG_1\otimes_1G_2(u, v) = dG_1(u)$,
(3) $dG_1\otimes_2G_2(u, v) = dG_1(u)\, dG_2^c(v)$,
(4) $dG_1\otimes_3G_2(u, v) = dG_2(v)$,
(5) $dG_1\otimes_4G_2(u, v) = dG_2^c(v)$,
(6) $dG_1\otimes_5G_2(u, v) = dG_1^c(u)\, dG_2(v)$,
(7) $dG_1\otimes_6G_2(u, v) = dG_1^c(u)$,
(8) $dG_1\otimes_7G_2(u, v) = dG_1^c(u)\, G_2^c(v)$

<u>Illustration of the above graph operations:</u>
Let us consider graphs G_1 and G_2 as following-

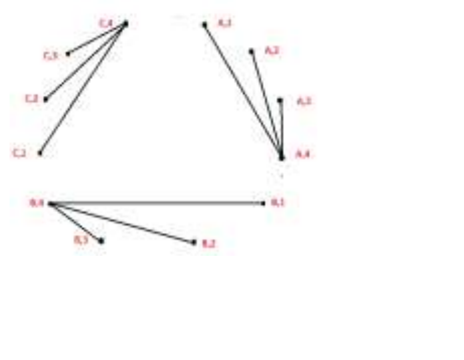
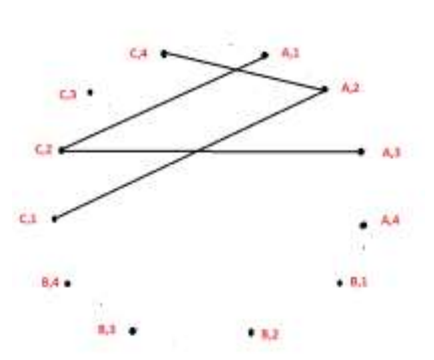
	
G_1	G_2

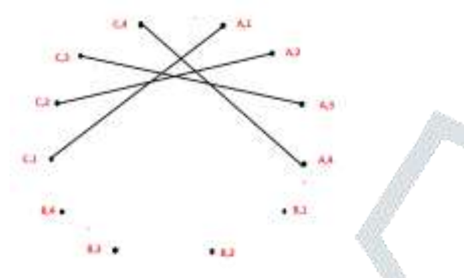
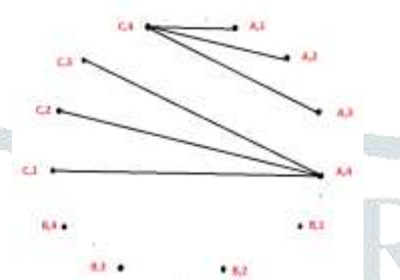
On applying the operations, we get the following graphs:

	
(i.) $G_1\otimes_0G_2$	(ii.) $G_1\otimes_1G_2$

	
(iii.) $G_1\otimes_2G_2$	(iv.) $G_1\otimes_3G_2$

[2] $dG_1\otimes_iG_2(u, v)$ = degree of vertex (u, v) of graph $G_1\otimes_iG_2$

	
(v.) $G_1 \otimes_4 G_2$	(vi.) $G_1 \otimes_5 G_2$

	
(vii.) $G_1 \otimes_6 G_2$	(viii.) $G_1 \otimes_7 G_2$

First Zagreb Index of New Graph Operations

In this section, we discuss the first Zagreb Index of new graphs and some of its special cases.[11]

Theorem 1: Let G_1^c, G_2^c be complement graphs of G_1 and G_2 respectively, Then

(1) $M_1(G_1 \otimes_0 G_2) = M_1(G_1)\, M_1(G_2)$

(2) $M_1(G_1 \otimes_1 G_2) = p_2 M_1(G_1)$

(3) $M_1(G_1 \otimes_2 G_2) = M_1(G_1)\, M_1(G_2^c)$

(4) $M_1(G_1 \otimes_3 G_2) = p_1 M_1(G_2)$

(5) $M_1(G_1 \otimes_4 G_2) = p_1 M_1(G_2^c)$

(6) $M_1(G_1 \otimes_5 G_2) = M_1(G_1^c) M_1(G_2)$

(7) $M_1(G_1 \otimes_6 G_2) = p_2 M_1(G_1^c)$

(8) $M_1(G_1 \otimes_7 G_2) = M_1(G_1^c) M_1(G_2^c)$

Proof: Using Lemma 1 and Lemma 2 and through the definition of first Zagreb index, we get

(1) $M_1(G_1 \otimes_0 G_2) = \sum_{(u,v)\in V((G1\otimes 0G2))} d^2_{(G1\otimes 0G2)}(u,v)$
$= \sum_{u\in V(G1)} \sum_{v\in V(G2)} d^2_{G1}(u)\, d^2_{G2}(v)$
$= \sum_{u\in V(G1)} d^2_{G1}(u) \sum_{v\in V(G_2^c)} d^2_{G2}(v)$
$= M_1(G_1)\, M_1(G_2)$

(2) $M_1(G_1 \otimes_1 G_2) = \sum_{(u,v)\in V((G1\otimes 1G2))} d^2_{(G1\otimes 1G2)}(u,v)$
$= \sum_{u\in V(G1)} d^2_{G1}(u) \sum_{v\in V(G2)} 1$
$= p_2 M_1(G_1)$

(3) $M_1(G_1 \otimes_2 G_2) = \sum_{(u,v)\in V((G1\otimes 2G2))} d^2_{(G1\otimes 2G2)}(u,v)$
$= \sum_{u\in V(G1)} \sum_{v\in V(G2)} d^2_{G1}(u)\, d^2_{G_2^c}(v)$
$= \sum_{u\in V(G1)} d^2_{G1}(u) \sum_{v\in V(G_2^c)} d^2_{G_2^c}(v)$
$= M_1(G_1)\, M_1(G_2^c)$

(4) $M_1(G_1 \otimes_3 G_2) = \sum_{(u,v)\in V((G1\otimes 4G2))} d^2_{(G1\otimes 4G2)}(u,v)$
$= \sum_{u\in V(G1)} 1 \sum_{v\in V(G2)} d^2_{G2}(v)$
$= p_1 M_1(G_2)$

(5) $M_1(G_1 \otimes_4 G_2) = \sum_{(u,v)\in V((G1\otimes 4G2))} d^2_{(G1\otimes 4G2)}(u,v)$
$= \sum_{u\in V(G1)} \sum_{v\in V(G2)} d^2_{G_1^c}(u)\, d^2_{G2}(v)$
$= p_1 M_1(G_2^c)$

(6) $M_1(G_1 \otimes_5 G_2) = \sum_{(u,v)\in V((G1\otimes 5G2))} d^2_{(G1\otimes 5G2)}(u,v)$
$= \sum_{u\in V(G1)} d^2_{G_1^c}(u) \sum_{v\in V(G2)} d^2_{G2}(v)$
$= M_1(G_1^c) M_1(G_2)$

(7) $M_1(G_1 \otimes_6 G_2) = \sum_{(u,v)\in V((G1\otimes 6G2))} d^2_{(G1\otimes 6G2)}(u,v)$
$= \sum_{u\in V(G1)} d^2_{G_1^c}(u) \sum_{v\in V(G2)} 1$
$= p_2 M_1(G_1^c)$

(8) $M_1(G_1 \otimes_7 G_2) = \sum_{(u,v)\in V((G1\otimes 7G2))} d^2_{(G1\otimes 7G2)}(u,v)$
$= \sum_{u\in V(G1)} \sum_{v\in V(G2)} d^2_{G_1^c}(u)\, d^2_{G_2^c}(v)$
$= M_1(G_1^c) M_1(G_2^c)$

Second Zagreb Index of New Graph Operations

In this section, we discuss the second Zagreb Index of new graphs.[11]

Theorem 2: Let G_1^c, G_2^c be complement graphs of G_1 and G_2 respectively, then
(1) $M_2(G_1 \otimes_0 G_2) = 2M_2(G_1)\, M_2(G_2)$,
(2) $M_2(G_1 \otimes_1 G_2) = p_2 M_2(G_1)$,
(3) $M_2(G_1 \otimes_2 G_2) = 2M_2(G_1)\, M_2(G_2^c)$,
(4) $M_2(G_1 \otimes_3 G_2) = p_1 M_2(G_2)$,
(5) $M_2(G_1 \otimes_4 G_2) = p_1 M_2(G_2^c)$,
(6) $M_2(G_1 \otimes_5 G_2) = 2M_2(G_1^c) M_2(G_2)$,
(7) $M_2(G_1 \otimes_6 G_2) = p_2 M_2(G_1^c)$,
(8) $M_2(G_1 \otimes_7 G_2) = 2M_2(G_1^c) M_2(G_2^c)$

Proof: By Lemma 1, Lemma 2 and definition of second Zagreb index we have,

(1) $M_2(G_1 \otimes_0 G_2) = \sum_{(a,b)\ (c,d)\in E(G1\otimes 0G2)} d_{(G1\otimes 0G2)}(a,b) d_{(G1\otimes 0G2)}(c,d)$
$= 2 \sum_{ac\in E(G1)} \sum_{bd\in E(G2)} d_{G1}(a) d_{G2}(b) d_{G1}(c) d_{G2}(d)$
$= \sum_{ac\in E(G1)} d_{G1}(a) d_{G1}(c) \sum_{bd\in E(G2)} d_{G2}(b) d_{G2}(d)$
$= 2M_2(G_1)\, M_2(G_2)$

(2) $M_2(G_1 \otimes_1 G_2) = \sum_{(a,b)\ (c,d)\in E(G1\otimes 1G2)} d_{(G1\otimes 1G2)}(a,b) d_{(G1\otimes 0G2)}(c,d)$
$= 2 \sum_{b=d=v\in V(G2)} \sum_{ac\in E(G1)} d_{G1}(a) d_{G2}(c)$
$= p_2 M_2(G_1)$

(3) $M_2(G_1 \otimes_2 G_2) = \sum_{(a,b)\ (c,d)\in E(G1\otimes 2G2)} d_{(G1\otimes 2G2)}(a,b) d_{(G1\otimes 2G2)}(c,d)$
$= 2 \sum_{ac\in E(G1)} \sum_{bd\in E(G2)} d_{G1}(a) d_{G_2^c}(b) d_{G1}(c) d_{G_2^c}(d)$
$= 2 \sum_{ac\in E(G1)} d_{G1}(a) d_{G1}(c) \sum_{bd\in E(G_2^c)} d_{G_2^c}(b) d_{G_2^c}(d)$
$= 2M_2(G_1)\, M_2(G_2^c)$

(4) $M_2(G_1 \otimes_3 G_2) = \sum_{(a,b)\ (c,d)\in E(G1\otimes 3G2)} d_{(G1\otimes 3G2)}(a,b) d_{(G1\otimes 3G2)}(c,d)$
$= \sum_{a=c=u\in V(G1)} \sum_{bd\in E(G2)} d_{G2}(b) d_{G2}(d)$
$= p_1 M_2(G_2)$

(5) $M_2(G_1 \otimes_4 G_2) = \sum_{(a,b)\ (c,d)\in E(G1\otimes 4G2)} d_{(G1\otimes 4G2)}(a,b) d_{(G1\otimes 4G2)}(c,d)$
$= \sum_{a=c=u\in V(G1)} (1) \sum_{bd\in E(G_2^c)} d_{G_2^c}(b) d_{G_2^c}(d)$

$= p_1 M_2(G_2^c)$

(6) $M_2(G_1 \otimes_5 G_2) = \sum_{(a,b)\,(c,d) \in E(G1 \otimes 5G2)} d_{(G1 \otimes 5G2)}(a,b) d_{(G1 \otimes 5G2)}(c,d)$
$= 2 \sum_{ac \in E(G_1^c)} \sum_{bd \in E(G2)} d_{G_1^c}(a) d_{G2}(b) d_{G_1^c}(c) d_{G2}(d)$
$= 2M_2(G_1^c) M_2(G_2)$

(7) $M_2(G_1 \otimes_6 G_2) = \sum_{(a,b)\,(c,d) \in E(G1 \otimes 6G2)} d_{(G1 \otimes 6G2)}(a,b) d_{(G1 \otimes 6G2)}(c,d)$
$= \sum_{b=d=v \in V(G2)} \sum_{ac \in E(G1)} d_{G_1^c}(a) d_{G2}(c)$
$= p_2 M_2(G_1^c)$

(8) $M_2(G_1 \otimes_7 G_2) = \sum_{(a,b)\,(c,d) \in E(G1 \otimes 7G2)} d_{(G1 \otimes 7G2)}(a,b) d_{(G1 \otimes 7G2)}(c,d)$
$= 2 \sum_{ac \in E(G_1^c)} \sum_{bd \in E(G_2^c)} d_{G_1^c}(a) d_{G_2^c}(b) d_{G_1^c}(c) d_{G_2^c}(d)$
$= \sum_{ac \in E(G_1^c)} d_{G_1^c}(a) d_{G_2^c}(c) \sum_{bd \in E(G_2^c)} d_{G_1^c}(b) d_{G_2^c}(d)$
$= 2M_2(G_1^c) M_2(G_2^c)$

Gourava Index of New Graph Operations

In this section, we study the first Gourava Index of new graphs and some of its special cases.

Theorem 3: Let G_1^c, G_2^c be complement graphs of G_1 and G_2 respectively, then
(1) $GO_1(G_1 \otimes_0 G_2) = M_1(G_1)\, M_1(G_2) + 2M_2(G_1)\, M_2(G_2)$,
(2) $GO_1\,(G_1 \otimes_1 G_2) = p_2\, GO_1\,(G_1)$,
(3) $GO_1\,(G_1 \otimes_2 G_2) = 2M_1(G_2)\, M_1(G_2^c) + 2M_2\,(G_2)\, M_2(G_2^c)$
(4) $GO_1\,(G_1 \otimes_3 G_2) = p_1\, GO_1\,(G_2)$,
(5) $GO_1\,(G_1 \otimes_4 G_2) = p_1\, GO_1\,(G_2^c)$,
(6) $GO_1\,(G_1 \otimes_5 G_2) = M_1(G_1^c) M_1(G_2) + 2M_2(G_1^c) M_2(G_2)$,
(7) $GO_1\,(G_1 \otimes_6 G_2) = p_2\, GO_1\,(G_1^c)$,
(8) $GO_1\,(G_1 \otimes_7 G_2) = M_1(G_1^c) M_1(G_2^c) + 2M_2(G_1^c) M_2(G_2^c)$

Proof: According to the definition $GO_1\,(G) = M_1(G) + M_2(G)$

(1) $GO_1(G_1 \otimes_0 G_2) = M_1(G_1 \otimes_0 G_2) + M_2\,(G_1 \otimes_0 G_2)$
$= M_1(G_1)\, M_1(G_2) + 2M_2(G_1)\, M_2(G_2)$

(2) $GO_1(G_1 \otimes_1 G_2) = M_1(G_1 \otimes_1 G_2) + M_2\,(G_1 \otimes_1 G_2)$
$= p_2 M_1(G_1) + p_2 M_2(G_1)$
$= p_2\,[(M_1(G_1) + M_2(G_1)]$
$= p_2\, GO_1\,(G_1)$

(3) $GO_1(G_1 \otimes_2 G_2) = M_1(G_1 \otimes_2 G_2) + M_2\,(G_1 \otimes_2 G_2)$
$= 2M_1(G_2)\, M_1(G_2^c) + 2M_2\,(G_2)\, M_2(G_2^c)$

(4) $GO_1(G_1 \otimes_3 G_2) = M_1(G_1 \otimes_3 G_2) + M_2\,(G_1 \otimes_3 G_2)$
$= p_1 M_1(G_2) + p_1 M_2(G_2)$
$= p_1\,(M_1(G_2) + M_2(G_2))$
$= p_1\, GO_1\,(G_2)$

(5) $GO_1(G_1 \otimes_4 G_2) = M_1(G_1 \otimes_4 G_2) + M_2\,(G_1 \otimes_4 G_2)$
$= p_1 M_1(G_2^c) + p_1 M_2(G_2^c)$
$= p_1\,[M_1(G_2^c) + M_2(G_2^c)]$
$= p_1\, GO_1\,(G_2^c)$

(6) $GO_1(G_1 \otimes_5 G_2) = M_1(G_1 \otimes_5 G_2) + M_2\,(G_1 \otimes_5 G_2)$
$= M_1(G_1^c) M_1(G_2) + 2M_2(G_1^c) M_2(G_2)$

(7) $GO_1(G_1 \otimes_6 G_2) = M_1(G_1 \otimes_6 G_2) + M_2\,(G_1 \otimes_6 G_2)$
$= p_2 M_1(G_1^c) + p_2 M_2(G_1^c)$
$= p_2\,[M_1(G_1^c) + p_2 M_2(G_1^c)]$
$= p_2\, GO_1\,(G_2^c)$

$$(8)\ GO_1(G_1 \otimes_7 G_2) = M_1(G_1 \otimes_7 G_2) + M_2(G_1 \otimes_7 G_2)$$
$$= M_1(G_1^c)M_1(G_2^c) + 2M_2(G_1^c)M_2(G_2^c)$$

In this section, we study the Second Gourava Index of new graphs and some of its special cases.

Theorem 4: Let G_1^c, G_2^c be complement graphs of G_1 and G_2 respectively, then

(1) $GO_2(G_1 \otimes_0 G_2) = 2\,GO_2(G_1)\,GO_2(G_2)$,
(2) $GO_2(G_1 \otimes_1 G_2) = p_2^2 GO_2(G_1)$,
(3) $GO_2(G_1 \otimes_2 G_2) = 2\,GO_2(G_1)\,GO_2(G_2^c)$,
(4) $GO_2(G_1 \otimes_3 G_2) = p_1^2 GO_2(G_2)$,
(5) $GO_2(G_1 \otimes_4 G_2) = p_1^2 GO_2(G_2^c)$,
(6) $GO_2(G_1 \otimes_5 G_2) = 2\,GO_2(G_1^c)\,GO_2(G_2)$,
(7) $GO_2(G_1 \otimes_6 G_2) = p_2^2 GO_2(G_1^c)$,
(8) $GO_2(G_1 \otimes_7 G_2) = 2\,GO_2(G_1^c)\,GO_2(G_2^c)$

Proof: According to the definition $GO_2(G) = M_1(G).M_2(G)$

$$(1)\ GO_2(G_1 \otimes_0 G_2) = M_1(G_1 \otimes_0 G_2).\ M_2(G_1 \otimes_0 G_2)$$
$$= M_1(G_1)\ M_1(G_2).\ 2M_2(G_1)\ M_2(G_2)$$
$$= 2\,GO_2(G_1)\,GO_2(G_2)$$

$$(2)\ GO_2(G_1 \otimes_1 G_2) = M_1(G_1 \otimes_1 G_2).\ M_2(G_1 \otimes_1 G_2)$$
$$= p_2 M_1(G_1).\ p_2 M_2(G_1)$$
$$= p_2^2\ [(M_1(G_1).\ M_2(G_1)]$$
$$= p_2^2 GO_2(G_1)$$

$$(3)\ GO_2(G_1 \otimes_2 G_2) = M_1(G_1 \otimes_2 G_2).\ M_2(G_1 \otimes_2 G_2)$$
$$= M_1(G_1)\ M_1(G_2^c).\ 2M_2(G_1)\ M_2(G_2^c)$$
$$= 2\,GO_2(G_1)\,GO_2(G_2^c)$$

$$(4)\ GO_2(G_1 \otimes_3 G_2) = M_1(G_1 \otimes_3 G_2).\ M_2(G_1 \otimes_3 G_2)$$
$$= p_1 M_1(G_2).\ p_1 M_2(G_2)$$
$$= p_1^2 GO_2(G_2)$$

$$(5)\ GO_2(G_1 \otimes_4 G_2) = M_1(G_1 \otimes_4 G_2).\ M_2(G_1 \otimes_4 G_2)$$
$$= p_1 M_1(G_2^c).\ p_1 M_2(G_2^c)$$
$$= p_1^2 GO_2(G_2^c)$$

$$(6)\ GO_2(G_1 \otimes_5 G_2) = M_1(G_1 \otimes_5 G_2).\ M_2(G_1 \otimes_5 G_2)$$
$$= M_1(G_1^c)M_1(G_2).2M_2(G_1^c)M_2(G_2)$$
$$= 2\,GO_2(G_1^c)\,GO_2(G_2)$$

$$(7)\ GO_2(G_1 \otimes_6 G_2) = M_1(G_1 \otimes_6 G_2).\ M_2(G_1 \otimes_6 G_2)$$
$$= p_2 M_1(G_1^c).\ p_2 M_2(G_1^c)$$
$$= p_2^2\ [M_1(G_1^c)\ .M_2(G_1^c)]$$
$$= p_2^2\ GO_2(G_2^c)$$

$$(8)\ GO_2(G_1 \otimes_7 G_2) = M_1(G_1 \otimes_7 G_2).\ M_2(G_1 \otimes_7 G_2)$$
$$= M_1(G_1^c)M_1(G_2^c).\ 2M_2(G_1^c)M_2(G_2^c)$$
$$= 2\,GO_2(G_1^c)\,GO_2(G_2^c)$$

Corollary 1. Let $G_1 \simeq C_n$ be a cycle graph with n vertices and $G_2 \simeq K_m$ be a complete graph with m vertices where $|V(C_n)| = n$, $|E(C_n)| = n$ and $|V(K_m)| = m$ $|E(K)| = \frac{m(m-1)}{2}$, then using theorem 3 we have,

(1) $GO_1(C_n \otimes_0 K_m) = 4nm^2(m-1)^2$
(2) $GO_1(C_n \otimes_1 K_m) = 8nm$
(3) $GO_1(C_n \otimes_2 K_m) = 0$
(4) $GO_1(C_n \otimes_3 K_m) = \frac{mn\,(m+1)}{2}$

(5) $GO_1(C_n \otimes_4 K_m) = 0$

(6) $GO_1(C_n \otimes_5 K_m) = mn(n-3)^2(m-1)^2(\frac{2+(n-3)(m-1)}{2})$

(7) $GO_1(C_n \otimes_6 K_m) = m\ (n(n-3)^2 + \frac{n}{2}\ (n-3)^3$

(8) $GO_1(C_n \otimes_7 K_m) = 0$

Proof: We know $|V(C_n)|= n$, $|E(C_n)|= n$ and $|V(K_m)| = m$ $|E(K_m)| = \frac{m(m-1)}{2}$

For C_n:

$|V(C_n)|=n$, $|E(C_n)|=n$, degree of each vertex is 2. then,

$M_1(C_n) = n\ (2+2) = 4n$

$M_2(C_n) = n\ (2.2) = 4n$

For C_n^c

We Know $C_n \cup C_n^c = K_n$

⇨ $|V(C_n^c)|=n$ and $|E(C_n^c)|= |E(K_n)| - |E(C_n)| = \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$ and

⇨ degree of each vertex of C_n^c = degree of each vertex of K_n – degree of each vertex of C_n

= (n-1) – 2 = n-3

So $M_1(C_n^c) = \frac{n(n-3)}{2}\ \big((n-3) + (n-3)\big)$

$= n(n-3)^2$

$M_2(C_n^c) = \frac{n(n-3)}{2}\ \big((n-3).(n-3)\big)$

$= \frac{n}{2}\ (n-3)^3$

For K_m:

$|V(K_m)|=m$, $|E(K_m)|= \frac{m(m-1)}{2}$, degree of each vertex is (m-1), then,

$M_1(K_m) = \frac{m(m-1)}{2}\ ((m\text{-}1) + (m\text{-}1)) = m(m-1)^2$

$M_2(K_m) = \frac{m(m-1)}{2}((m\text{-}1).(m\text{-}1)) = \frac{m}{2}\ (m-1)^3$

For K_m^c:

We Know $K_m \cup K_m^c = K_m$

⇨ $|V\ (K_m^c)|=m$ and $|E\ (K_m^c)|= |E(K_m)| - |E(K_m)| = 0$ and

⇨ degree of each vertex of K_m^c = degree of each vertex of K_m – degree of each vertex of K_m

= 0

So $M_1(K_m^c) = \frac{m(m-1)}{2}\ (0+0)$

$= 0$

$M_2(C_n^c) = \frac{m(m-1)}{2}\ (0.0)$

$=0$

Therefore,

(1) $GO_1(C_n \otimes_0 K_m) = M_1(C_n)\ M_1(K_m) + 2M_2\ (C_n)\ M_2(K_m)$

$= 4nm(m-1)^2 + 2\ (4n)\ \frac{m(m-1)^3}{2}$

$= 4nm^2(m-1)^2$

(2) $GO_1(C_n \otimes_1 K_m) = p_2\ GO_1\ (C_n)$

$= p_2\ (M_1(C_n) + M_2(C_n))$

$= m\ (4n + 4n)$

$= 8nm$

(3) $GO_1(C_n \otimes_2 K_m) = 2M_1(K_m)\ M_1(K_m^c) + 2M_2(K_m)\ M_2(K_m^c)$

$= 2\ m(m-1)^2.0 + 2\frac{m}{2}\ (m-1)^3.0$

$= 0$

(4) $GO_1(C_n \otimes_3 K_m) = p_1\, GO_1\,(K_m)$
$= p_1\,(\,M_1(K_m) + M_2(K_m))$
$= n\,(m(m-1)^2 + \frac{m}{2}\,(m-1)^3)$
$= n\,m(m-1)^2[\,1+\frac{1}{2}\,(m-1)^1)$
$= \frac{1}{2}mn\,(m+1)$

(5) $GO_1(C_n \otimes_4 K_m) = p_1\, GO_1\,(K_m^c)$
$= p_1\,(M_1\,(K_m^c) + M_2(K_m^c))$
$= n\,(0+0)$
$= 0$

(6) $GO_1(C_n \otimes_5 K_m) = M_1(C_n^c)M_1(K_m) + 2M_2(C_n^c)M_2(K_m)$
$= n(n-3)^2\, m(m-1)^2 + 2\frac{n}{2}\,(n-3)^3\,\frac{m}{2}\,(m-1)^3$
$= mn(n-3)^2(m-1)^2(\frac{2+(n-3)(m-1)}{2})$

(7) $GO_1(C_n \otimes_6 K_m) = p_2\, GO_1\,(C_n^c)$
$= p_2\,(M_1\,(C_n^c) + M_2(C_n^c))$
$= m\,(n(n-3)^2 + \frac{n}{2}\,(n-3)^3)$

(8) $GO_1(C_n \otimes_7 K_m) = M_1(C_n^c)M_1(K_m^c) + 2M_2(C_n^c)M_2(K_m^c)$
$= M_1(C_n^c).0 + 2M_2(C_n^c).0$
$= 0$

Corollary 2. Let $G_1 \simeq C_n$ be a cycle graph with n vertices and $G_2 \simeq K_m$ be a complete graph with m vertices, where $|V(C_n)|=n$, $|E(C_n)|=n$ and $|V(K_m)|= m$ $|E(K)|= \frac{m(m-1)}{2}$, then using theorem 4 we have,
(1) $GO_2(C_n \otimes_0 K_m) = 16n^2m^2(m-1)^2$,
(2) $GO_2(C_n \otimes_1 K_m) = 16m^2n^2$,
(3) $GO_2(C_n \otimes_2 K_m) = 0$,
(4) $GO_2(C_n \otimes_3 K_m) = \frac{m^2n^2(m-1)^5}{2}$
(5) $GO_2(C_n \otimes_4 K_m) = 0$
(6) $GO_2(C_n \otimes_5 K_m) = n^2(n-3)^2m^2(m-1)^5$,
(7) $GO_2(C_n \otimes_6 K_m) = \frac{m^2n^2(n-3)^5}{2}$
(8) $GO_2(C_n \otimes_7 K_m) = 0$

Proof:

(1) $GO_2(C_n \otimes_0 K_m) = 2\, GO_2\,(C_n)\, GO_2\,(K_m)$
$= 2\,(M_1\,(C_n)\, M_2\,(C_n))\,(M_1\,(K_m)\, M_2\,(K_m)\,)$
$= 2(4n)\,(4n)\,(m(m-1)^2\,\frac{m}{2}\,(m-1)^3)$
$= 16n^2m^2(m-1)^2$

(2.) $GO_2(C_n \otimes_1 K_m) = p_2^2 GO_2\,(C_n)$
$= p_2^2\,(M_1(C_n).M_2(C_n))$
$= m^2\,(4n.4n)$
$= 16m^2n^2$

(3.) $GO_2(C_n \otimes_3 K_m) = 2\, GO_2\,(C_n)\, GO_2\,(K_m^c)$,
$= 2\,(M_1(C_n)\, M_2\,(C_n))\,(M_1\,(K_m^c)\, M_2\,(K_m^c))$
$= 2\,(4n)\,(4n)\,(0)(0)$
$= 0$

(4.) $GO_2(C_n \otimes_3 K_m) = p_1^2\ GO_2\ (K_m)$

$= p_1^2\ (M_1(K_m).M_2(K_m))$

$= n^2\ ((m(m-1)^2.\frac{m}{2}\ (m-1)^3)$

$= \frac{m^2n^2(m-1)^5}{2}$

(5.) $GO_2(C_n \otimes_4 K_m) = p_1^2 GO_2\ (K_m^c)$

$= p_1^2\ (M_1\ (K_m^c).M_2(K_m^c))$

$= n^2\ (0+0)$

$= 0$

(6.) $GO_2(C_n \otimes_5 K_m) = 2\ GO_2\ (C_n^c)\ GO_2\ (K_m)$

$= 2\ (M_1(C_n^c)\ M_2\ (C_n^c))\ (M_1\ (K_m)\ M_2\ (K_m))$

$= 2\ n(n-3)^2 \frac{n}{2}\ (n-3)^3\ m(m-1)^2 \frac{m}{2}\ (m-1)^3$

$= \frac{m^2n^2(n-3)^5(m-3)^5}{2}$

(7.) $GO_2(C_n \otimes_6 K_m) = p_2^2 GO_2\ (K_m^c)$

$= p_2^2\ (M_1\ (K_m^c).\ M_2(K_m^c))$

$= n^2\ (0.0)$

$= 0$

(8.) $GO_2(C_n \otimes_7 K_m) = 2\ GO_2\ (G_1^c)\ GO_2\ (G_2^c)$

$= 2\ (M_1\ (C_n^c)\ M_2\ (C_n^c))\ (M_1\ (K_m^c)\ M_2\ (K_m^c))$

$= 2\ n(n-3)^2 \frac{n}{2}\ (n-3)^3(0.0)$

$= 0$

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