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The Embedded Mathematics Behindhand Differential Calculus

Author:

Pallavi Saikia.
Former Assistant Professor,
Department of Mathematics,
Kokrajhar Government College,

Abstract:

Calculus is the extensively pertinent sub- sectioning of Mathematics that portrays the precise exploration of uninterrupted variation. The sub-section Calculus possesses two foremost shrubs viz., differential calculus and integral calculus where differential calculus apprehended the instantaneous frequencies of alteration of the gradients of curves, whereas the sub-branch integral calculus concentrates about the accrual of extents and calculation of areas bounded by the co-ordinate axes and the curve. The two sub-arenas of Calculus viz., Differential calculus and Integral calculus are associated by means of the fundamental theorem of Calculus which statuses that differentiation is the reverse procedure to integration. These two branches are correlated to one another by the fundamental theorem of calculus and they branded the practice of the fundamental concepts of convergence of infinite sequences and infinite series corresponding conceptualizing the concept of a satisfactorily -demarcated limit. In Mathematics, differential calculus is a sub arena of Calculus that comprehensions the frequencies at which measures or expanse variates. It is one of the two long-established dissections of the subject matter Calculus whereas the second one is integral calculus—the analysis of the expanse underneath a curve. Differential Calculus is that branch of mathematical analysis, devised by Isaac Newton and G.W. Leibniz It is concerned with the problem of finding the rate of change of a function with respect to the variable on which it depends. Thus, it involves calculating derivatives and using them to solve problems involving nonconstant rates of change. Typical applications include finding maximum and minimum values of functions in order to solve practical problems in optimization.

Abstract: calculus, derivative, differentiation, L’Hôpital’s rule, quotient rule.

1.Introduction:

It was the last part of the 17th century in which the subject topic, Calculus of indiscernible deviation procured adequate nourishment to technologically progress itself with the significant involvements ensured analytically by two prodigious protuberant characters viz., Sir Isaac Newton and Gottfried Wilhelm Leibniz. In today’s epoch, the topic Calculus has prevalent utilizations in solving scientific, engineering problems and complicacies arising in social science disciplines. The chief substances of edifying in the sub-arena

Differential Calculus lies in finding the derivatives of a function portraying synonyms such as differentials and their practical implicative appliances. The derivative of a function at a preferred input value designates the rate of change of the function adjacent to the input value. The procedure deployed in determining a derivative is termed as the process of differentiation. Geometrically, the derivative at a point is the gradient of the tangent line drawn to the graph of the function at that prescribed point, resting on the condition that the derivative of the function endures and is well-defined at that particular point. For a real-valued function of a single real variable, the derivative of a function at a point usually regulates the finest linear estimate to the function at that particular point.

The efficient procedure of employing differentiation of various functions deploys its applications in approximately all quantitative disciplines. In Physics, the process of finding the derivative of the displacement of a traversing entity in correspondence with time is the velocity of the entity and the finding the derivative of the velocity with respect to time is acceleration. The derivative of the momentum of a body with respect to time generates the force exerted on the body, reshuffling the very derivative statement conforms to the well-known law of Physics as established by the great physicist Sie Isaac Newton or more commonly marke as Newton's Second Law of Motion as $\mathbf{F} = m\mathbf{a}$. We can assume some practical scenarios based on the differentiation criteria as the rate of change of a chemical reaction is a derivative, in Operations Research, derivatives usually regulate the most effective customs to conveyance constituents and enterprise workshops.

Derivatives are recurrently cast-off to treasure the maxima and minima of a function. Equations concerning derivatives are called differential equations and are essential in unfolding phenomena around our observatory universe. Derivatives and their simplifications give the impression in countless grounds of the subject Mathematics like complex analysis, measure theory, differential geometry, functional analysis and abstract algebra.

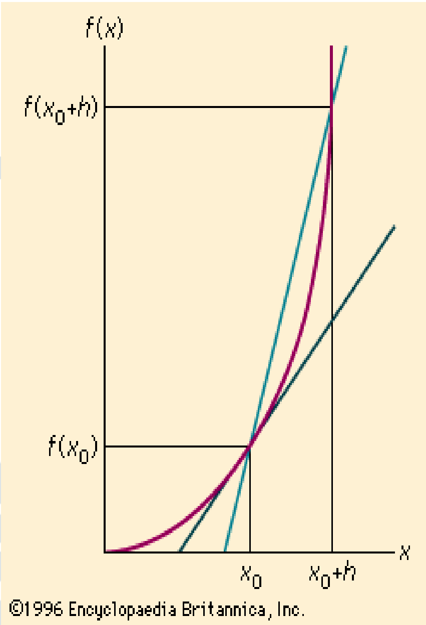
1.1 Fundamental theorem of calculus:

Fundamental theorem of calculus is the rudimentary determinant of the Calculus. It narrates the derivative to the integral and makes available the prime technique of appraising definite integrals. In short-term ephemeral, it circumstances that any continuous function over an interval has an antiderivative i.e., a function whose frequency of variation or evaluating derivatives, equals the function on that particular interval. Supplementarily, the definite integral of such a function over an interval designated by the inequality $a < x < b$ is the difference of the two functional values i.e., $F(b) - F(a)$, where F is an antiderivative of the prescribed function. This predominantly sophisticated theorem demonstrates the inverse function association of the derivative and the integral and assists as the mainstay of the tangible sciences. It was expressed self-sufficiently by Isaac Newton and Gottfried Wilhelm Leibniz without knowing one another.

1.2 Derivative

In Mathematics, the process of finding the derivative of a function lies on the principle of enumerating the rate of alteration of a function with respect to a particular variable to be differentiated. Derivatives are essential in finding the solution of problems in Calculus and Differential Equations. In convention, scientists perceive fluctuating arrangements e.g., dynamical systems or pre-arrangements to acquire the degree of variation of approximately a number of variables of curiosity, integrate this evidential statistical data into a good quantity of differential equations and usage of integration techniques to attain a function that can be castoff to envisage the behavioural characteristics of the innovative structure underneath some existing varied circumstances.

The geometrical explanation of the derivative of a function can be mathematically abridged as the derivative of a function can be deciphered as the slope or gradient of the graph of the function or additionally assuredly as the slope or gradient of the tangent line drawn at a point. Its computation under practical scenario, springs from the gradient formula established for a straight line, excluding that a limiting procedure must be cast-off for curvatures. The gradient is very frequently articulated or in Cartesian Co-ordinate systems, it has been presented as the proportion of the alteration or variation in *y* co-ordinates corresponding to the change in *x* co-ordinates. For the straight line shown in the figure, the formula for the slope is established $(y_1 - y_0)/(x_1 - x_0)$. Additional technique to prompt this formula is $[f(x_0 + h) - f(x_0)]/h$, where the very term *h* stands for $x_1 - x_0$ and the function of f over the variable x is f(x) for y i.e., the in terms of equation it can be presented as y = f(x). This change in notation is useful for advancing from the idea of the slope of a line to the more general concept of the derivative of a function.

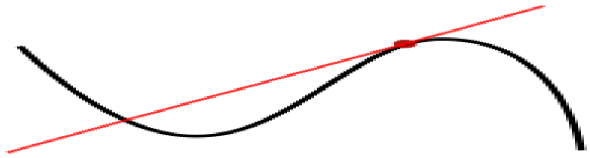


For a curvature, the very ratio be contingent on where the points are preferred, revealing the statistic that curves do not have a constant slope. To find out the slope or gradient at an estimated point, the selection of the second point desirable to calculate the ratio represents a trouble since over-all, the proportion will characterize only an average slope flanked by the points, relatively than the actual slope at either point as depicted in the adjoining figure. To get around this difficulty, a limiting procedure is cast-off where by the second point is not fixed but specified by a variable, as length or height wherever applicable, *h* in the ratio for the straight line above. Finding the limit in this circumstance is a progression of finding a number that the ratio approaches as *h* approaches 0 so that the limiting ratio will represent the actual slope at the given point. Some manipulations must be done on the quotient $[f(x_0 + h) - f(x_0)]/h$ so that it can be redrafted in a form in which the limit as *h* approaches 0 can be perceived more directly. For instance, we can consider the parabola given by $y=x^2$. In finding the derivative of x^2, the quotient is $[(2 + h)^2 - 2^2]/h$. By expanding the numerator, the quotient becomes $(4 + 4h + h^2 - 4)/h = (4h + h^2)/h$. Both numerator and denominator still move towards 0 but if *h* is not actually zero but only very close to it then *h* can be partitioned out, giving $4 + h$ which is effortlessly perceived to approach 4 as *h* approaches 0.

Figure 1 showing the slope of a curve

Applying summing up, the derivative of $f(x)$ at x_0, written as $f'(x_0)$, $(df/dx)(x_0)$ or $Df(x_0)$, is defined as $\lim_{h \to 0} [f(x_0 + h) - f(x_0)]/h$ if this limit exists.

The process of differentiation i.e., manipulating the derivative, infrequently necessitates the usage or practice of the elementary definition, as an alternative can be proficient through an acquaintance of the three elementary derivatives, the use of four instructions of operation and a knowledge of how to calculate functions.



The figure 1 deploys the graph of a function, sketched in black and have traced a tangent line to that function, coloured in red. The slope or gradient of the tangent line equivalents the derivative of the function at the perceptible point.

1.3 The Underlying Procedure of Differentiation:

In Mathematics, differentiation is the procedure of finding the derivative or frequency of alteration of a function. In divergence to the nonconcrete status of the theory behindhand it, the applied procedure of practice of differentiation can be supported rigidly by virtuously algebraic manipulations, using three basic derivatives, four rules of operation, and a knowledge of how to manipulate functions.

The three basic derivatives D are:

- In case of algebraic functions, the derivative can be expressed as $D(x^n) = nx^{n-1}$, where n is any real number.
- In case of trigonometric functions like sinx and cosx, the respective formulae for calculating the derivatives of theses two trigonometrical functions are as follows:
  $D(\sin x) = \cos x$ and $D(\cos x) = -\sin x$ and
- In case of exponential functions, the derivative can be found as $D(e^x) = e^x$.

For those varieties of functions which are the outcomes of the of amalgamations of these classes of functions, the concept offers the following uncomplicated instructions for differentiating the sum, product or quotient of any two functions such as $f(x)$ and $g(x)$ of the variable x, the derivatives are as follows, by considering a and b as constants, we have the derivatives here as: $D(a\,f + b\,g) = a\,Df + b\,Dg$ (sums), $D(f.g) = fDg + gDf$ (products) and $D(f/g) = (gDf - fDg)/g^2$ (quotients).

The supplementary straightforward rule, so-called the chain rule, delivers a technique to differentiate a composite function. Thus, if $f(x)$ and $g(x)$ are two functions, the composite function $f(g(x))$ is calculated for a value of x by first evaluating $g(x)$ and then evaluating the function f at this value of $g(x)$; for instance, if $f(x) = \sin x$ and $g(x) = x^2$, then $f(g(x)) = \sin x^2$, while $g(f(x)) = (\sin x)^2$. The chain rule statuses that the derivative of a composite function is given by a product, as $D(f(g(x))) = D\,f(g(x)) \cdot Dg(x)$. In languages, the first factor on the right, $D\,f(g(x))$, indicates that the derivative of $D\,f(x)$ is first found as usual, and then x wherever it occurs is replaced by the function $g(x)$. For instance, in case of the trigonometric function, $\sin x^2$, the Chain rule progresses the calculation in terms of applying Chain rule as $D(\sin x^2) = D\sin(x^2) \cdot D(x^2) = (\cos x^2) \cdot 2x$.

According to the German Mathematician Gottfried Wilhelm Leibniz's representation, which uses d/dx in place of D and thus agrees differentiation with respect to dissimilar variables to be constructed explicitly, the chain rule takes the more unforgettable symbolic cancellation form: $d(f(g(x)))/dx = df/dg \cdot dg/dx$.

2. Depiction of Review of Related Literature in the Arena of Differentiation Process:

The perception of a derivative in the intelligence of a tangent line is an actual old one which is accustomed to earliest Greek Mathematicians. In this context, we can utter the name of the great Mathematician Euclid in 300 BC, Archimedes, ranging between 287–212 BC and Apollonius of Perga, during the domain ranges between the periods viz., 262–190 BC. Archimedes also perceived usage of indivisibles, even though these were principally used to analyze areas and volumes moderately than derivatives and tangents.

The use of infinitesimals to study the frequencies of change can be instituted in Indian Mathematics, conceivably as early as 500 AD, when the astronomer and mathematician Aryabhata during 476–550 BC, used infinitesimals to study the trajectory of the Moon. The use of infinitesimals to calculate rates of change was technologically advanced significantly by Bhāskara II during the times from 1114–1185 BC. Undeniably, it has been claimed that countless of the significant concepts of differential calculus can be instituted in his work, such as Rolle's Mean Value Theorem.

The Mathematician, Sharaf al-Dīn al-Tūsī , during the epoch of History,1135–1213 BC, depicted in his creation Treatise on Equations, recognized circumstances for some cubic equations to have solutions, by the means of calculating the maxima of suitable cubic polynomials. We can obtain for instance, the maximum for having positive value of the variable x of the cubic polynomial $ax^2 - x^3$ transpires when $x = 2a / 3$ and clinched thereof that the equation $ax^2 = x^3 + c$ has exactly one positive solution when considered the value of c as $c = 4a^3 / 27$ and two consecutive positive solutions whenever $0 < c < 4a^3 / 27$. The historiographer of science, Roshdi Rashed, has reasoned that al-Tūsī must have used the derivative of the cubic to attain this result. Rashed's inference has been challenged by other scholars, nevertheless, who claim that he could have obtained the result by additional approaches which do not necessitate the derivative of the function to be acknowledged.

The up-to-the-minute expansion of calculus is frequently accredited to Isaac Newton (1643–1727) and Gottfried Wilhelm Leibniz (1646–1716), who delivered independent and amalgamated tactics to differentiation and derivatives. The significant understanding, nevertheless that produced them this acknowledgement, was the fundamental theorem of calculus, connecting differentiation and integration, this reduced old-fashioned, greatest earlier methods for calculating areas and volumes which had not been suggestively protracted since the time of Ibn al-Haytham (Alhazen). For their philosophies on derivatives, both Newton and Leibniz constructed on important previous work by mathematicians such as Pierre de Fermat (1607-1665), Isaac Barrow (1630–1677), René Descartes (1596–1650), Christiaan Huygens (1629–1695), Blaise Pascal (1623–1662) and John Wallis (1616–1703). Regarding Fermat's influence, Newton was the first and foremost to spread over differentiation to theoretical physics, whereas Leibniz methodically established much of the symbolization that are still used today.

Subsequently from the 17th century, several mathematicians have subsidized to the theory of differentiation. In the 19th century, calculus was invented as a much additional demanding equilibrium by mathematicians such as Augustin Louis Cauchy during the periods 1789–1857, Bernhard Riemann in the years 1826–1866 and Karl Weierstrass from the periods 1815–1897. It was also during this period that the differentiation was comprehensive to Euclidean space and the complex plane simultaneously.

3. L’Hôpital’s rule

L’Hôpital’s rule, in Real as well as Complex Analysis, a wide practice of differential calculus for appraising indeterminate forms such as 0/0 and ∞/∞ when they consequence from an effort to treasure a limit. It is christened for the French mathematician Guillaume-François-Antoine, marquis de L’Hôpital, who acquired the formula from his teacher the Swiss mathematician Johann Bernoulli. L’Hôpital published the formula in

L'Analyse des infiniment petits pour l'intelligence des lignes courbes (1696), the very elementary textbook on differential calculus.

L'Hôpital's rule configures that when the limit of $f(x)/g(x)$ is indeterminate, underneath guaranteed conditions it can be attained by estimating the limit of the quotient of the derivatives of the functions f and g (i.e., $f'(x)/g'(x)$). If this outcome is indeterminate, the technique can be recurrent.

4. Quotient rule:

Quotient rule is the rule for finding the derivative of a quotient of two functions. If both f and g are differentiable, then so is the quotient $f(x)/g(x)$. In abbreviated notation, it says $(f/g)' = (gf' - fg')/g^2$.

The primary objects of study in differential calculus are the derivative of a function, related notions such as the differential, and their applications. The derivative of a function at a chosen input value describes the rate of change of the function near that input value. The process of finding a derivative is called **differentiation**. Geometrically, the derivative at a point is the slope of the tangent line to the graph of the function at that point, provided that the derivative exists and is defined at that point. For a real-valued function of a single real variable, the derivative of a function at a point generally determines the best linear approximation to the function at that point.

Differential calculus and integral calculus are connected by the fundamental theorem of calculus, which states that differentiation is the reverse process to integration.

Differentiation has applications in nearly all quantitative disciplines. In physics, the derivative of the displacement of a moving body with respect to time is the velocity of the body, and the derivative of the velocity with respect to time is acceleration. The derivative of the momentum of a body with respect to time equals the force applied to the body; rearranging this derivative statement leads to the famous $\mathbf{F} = m\mathbf{a}$ equation associated with Newton's second law of motion. The reaction rate of a chemical reaction is a derivative. In operations research, derivatives determine the most efficient ways to transport materials and design factories.

Derivatives are frequently used to find the maxima and minima of a function. Equations involving derivatives are called differential equations and are fundamental in describing natural phenomena. Derivatives and their generalizations appear in many fields of mathematics, such as complex analysis, functional analysis, differential geometry, measure theory, and abstract algebra.

5. The Embedded Concept Behind Evaluating a Derivative:

However, many functions cannot be differentiated as easily as polynomial functions, meaning that sometimes further techniques are needed to find the derivative of a function. These techniques include the chain rule, product rule, and quotient rule. Other functions cannot be differentiated at all, giving rise to the concept of differentiability.

A closely related concept to the derivative of a function is its differential. When x and y are real variables, the derivative of f at x is the slope of the tangent line to the graph of f at x. Because the source and target of f are one-dimensional, the derivative of f is a real number. If x and y are vectors, then the best linear approximation to the graph of f depends on how f changes in several directions at once. Taking the best linear approximation in a single direction determines a partial derivative, which is usually denoted $\partial y/\partial x$. The linearization of f in all directions at once is called the total derivative.

6. Applications of derivatives:

6.1 Optimization:

If a function *f* is a differentiable function on the set of reals $\mathbb{R}$ in an open interval and *x* is a local maximum or a local minimum of function *f* then the derivative of the function *f* at *x* is zero. It is the points where $f'(x) = 0$ are called critical points or stationary points and the value of f at x is called a critical value. If f is not presumed to be differentiable everywhere, the points at which it becomes unsuccessful to be differentiable are correspondingly described as critical points.

If the very function *f* is differentiable twice, inversely, a critical point *x* of function *f* can be analysed by reproducing the second derivative of function *f* at the point *x*:

- if it is positive, the point constant *x* is a local minimum
- if it is negative, *x* is a local maximum.
- if it is zero, *x* could be a local minimum, a local maximum or neither. We can pre assume for illustration the function $f(x) = x^3$ which possess a critical point at $x = 0$ but has neither a maximum nor a minimum there, on the other hand, in case of the function $f(x) = \pm x^4$ has a critical point at $x = 0$ and a minimum and a maximum there correspondingly.

This is so-called the second derivative assessment. A substitute method, named the first derivative assessment, encompasses considering the sign of the derivative of the function f i.e., f' on each side of the critical point.

Captivating derivatives and resolution for critical points is consequently frequently a unpretentious technique to treasure local minima or maxima which can be advantageous in optimization. By the extreme value theorem, a continuous function on a closed interval must accomplish its minimum and maximum values at least once. If the function is differentiable, the minima and maxima can solitarily transpire at critical points or endpoints.

This correspondingly has demonstrations representing applications in graph tracing, once the local minima and maxima of a differentiable function have been calculated, a rough plot of the graph can be attained from the inspection that it will be either increasing or decreasing in between the critical points.

In sophisticated magnitudes, a critical point of a scalar valued function is a point at which the slope is calculated to be equal to the value zero. The second derivative examination can still be cast-off to analyse critical points by considering the eigenvalues of the Hessian matrix of second partial derivatives of the function at the critical point. If all of the eigenvalues are positive, then the point is a local minimum; if all are negative, it is a local maximum. If there are some positive and some negative eigenvalues, then the critical point is termed as a saddle point and if none of these cases persists, i.e., a few of the eigenvalues are zero then the test is well-thought-out to be unconvincing.

6.2 Calculus of variations:

An illustration of an optimization problem is to calculate the shortest curvature between two points on a surface, presumptuous that the curve must also be sketched on the same surface. If the surface is a plane one, the shortest curve drawn is a line. On the contrary, if the surface is, an oval-shaped, the smallest track traversed is not instantaneously distinct. These paths are called geodesics and one of the greatest vital difficulties in the calculus of variations is finding geodesics. Further illustration is when we are to find the smallest area surface stuffing in a closed curvature in space. This surface is so-called a minimal surface and it can also be found using the calculus of variations.

6.3 Applications of Derivatives in Physics:

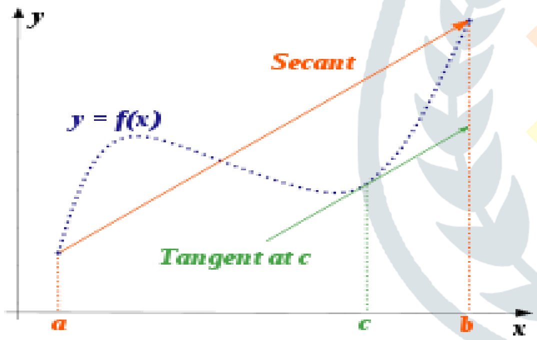
The theory of Calculus is of vital importance in Physics: numerous physical processes are designated by equations concerning derivatives, called differential equations. Physics is predominantly concerned with the technique with which quantities variates and progress over time and the perception of the **time derivative**, the frequency of variation in correspondence to time factor is indispensable for the specific description of numerous significant perceptions. Precisely, the time derivatives of an article's location are important in Newtonian Physics. We can have

- velocity is the derivative with respect to time of an entity's displacement which means distance from the unique allocation.
- acceleration is the derivative with respect to time of an article's velocity which means that the second derivative calculated with respect to time of an entity's location.

6.4 In Finding Out the Solutions of the Differential equations:

A differential equation is a relation between a assemblage of functions and their related derivatives. An ordinary differential equation can be defined as a differential equation that narrates functions of one variable to their respective derivatives with respect to that variable. A partial differential equation is a differential equation that associates functions of more than one variable to their partial derivatives. Differential equations ascend unsurprisingly in mathematical modelling, physical sciences and within mathematics itself. For illustration, Newton's second law which designates the relationship between acceleration and force applied can be specified as the ordinary differential equation.

6.5 Deployment in the Functioning of Mean value theorem:



The mean value theorem stretches an association flanked by values of the derivative and values of the original function. If $f(x)$ is a real-valued function and a and b are number constants with the inequality persistent condition $a < b$, under this circumstance, the mean value theorem enunciates that under some pre-assumed hypothesis, the slope or gradient between the two points $(a, f(a))$ and $(b, f(b))$ is identical to the slope of the tangent line drawn to the function f at some point c between the two numbers viz., a and b.

In repetition, what the mean value theorem does is control a function in terms of its derivative. For instance, suppose that f has derivative equal to zero at each point. This means that its tangent line is horizontal at every point, so the function should also be horizontal. The mean value theorem proves that this must be true: The slope between any two points on the graph of function f must equal the slope of one of the tangent lines of function f. All of those slopes are zero, so any line from one point on the graph to another point will also have slope zero. But that says that the function does not move up or down, so it must be a horizontal line. More complicated conditions on the derivative lead to less precise but still highly useful information about the original function.

6.6 Taylor polynomials and Taylor series:

The derivative stretches the unsurpassed conceivable linear estimate of a function at a specified point, but this can be very dissimilar from the original function. It is a technique of enlightening the approximation is to receipts a quadratic approximation. The linearization of a real-valued function $f(x)$ at some pre-assigned point x_0 is a linear polynomial, denoted by the algebraic expressions $a + b(x - x_0)$, and might be conceivable to acquire a improved assessment by reflecting a quadratic polynomial $a + b(x - x_0) + c(x - x_0)^2$. Further we can presume about a cubic polynomial $a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3$ and this knowledge can be prolonged to arbitrarily polynomials of higher degree. For each one of these polynomials, there should be a finest conceivable excellence of probable set of coefficients like a, b, c, and d that brands the consideration as virtuous as probable.

In the neighbourhood of x_0, for a the greatest imaginable choice is always $f(x_0)$ and for other constant quantity, b the best enumerable selection is often denoted mathematically by $f'(x_0)$. For the additional constants, c, d, and higher-degree coefficients, these coefficients are governed by higher derivatives of the product of the two functions f. c should always be $f''(x_0)/2$, and d should always be $f'''(x_0)/ 3!$ Using these coefficients gives the Taylor polynomial of the function f. The Taylor polynomial of degree d is the polynomial of degree d which best approximates f, and its coefficients can be found by a generalization of the above formulas. Taylor's theorem gives a precise bound on how good the approximation is. If f is a polynomial of degree less than or equal to d, then the Taylor polynomial of degree d equals f.

The limit of the Taylor polynomials is an infinite series entitled as the very familiar Taylor series. The Taylor series is recurrently an exceptionally upright estimation to the unique primordial function. Functions which are equivalent to their Taylor series are also designated with the name as analytic functions. It is incredible for functions with disjointedness or discontinuities, the sharp corners to be analytic, additionally there happen smooth functions which are not always analytic.

6.7 Implicit function theorem:

A good number of natural geometrical shapes like circles, cannot be sketched as the graph of a function. In this context, we can comprehend that if $f(x, y) = x^2 + y^2 - 1$, then the circle is the set of all ordered pairs, represented by (x, y) such that $f(x, y) = 0$. This set is entitled as the zero set of the function f and is not the identical as the graph of the function f which stands for a paraboloid. The implicit function theorem translates relations presented by the equation f $(x, y) = 0$ into functions. It circumstances that if f is continuously differentiable, around majority of the points, the zero set of the function f appearances like graphs of functions inserted together. The very significant points where this is not accurate are regulated by a condition on the derivative of function f. The case where circle, for illustration, can be attached together from the graphs of the two functions $\pm\sqrt{\boxed{1 - x^2}}$. In a neighbourhood of every point on the circle except the two co-ordinates viz., $(-1, 0)$ and $(1, 0)$, one of these two functions has a graph that appearances like the circle. Actually, these two functions also ensue to meet at the two points $(-1, 0)$ and $(1, 0)$, however this is not certain by the implicit function theorem.

The implicit function theorem is meticulously connected to the inverse function theorem which statuses when a function appearances like graphs of invertible functions inserted together.

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