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Approximative Atomic Decompositions in Banach Spaces

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Abstract: Approximative atomic decompositions in Banach spaces have been formally defined, and a characterization for these decompositions has been established. It has been proven that a Banach space E possesses an approximative atomic decomposition if and only if it satisfies the bounded approximation property. Additionally, sufficient conditions for the existence of approximative atomic decompositions in separable Banach spaces have been identified. As an application of these decompositions, it has been demonstrated that if both E and F are Banach spaces with the bounded approximation property, then their Cartesian product E×F also possesses this property.

1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [3] while addressing significant challenges in non-harmonic Fourier series. Feichtinger and Grochenig [6] extended the concept of frames to Banach spaces and termed them atomic decompositions.Atomic decompositions have been pivotal in the advancement of wavelet theory and Gabor theory. Frazier and Jawerth [7] constructed wavelet atomic decompositions, known as ϕ-transforms, for Besov spaces. Feichtinger [5] developed Gabor atomic decompositions for modulation spaces, which are Banach spaces similar to Besov spaces, characterized by conditions of smoothness and decay.Walnut [13] extended many results to weighted (L^2) spaces, demonstrating that Gabor atomic decompositions in ($L^2_w(R)$ may not necessarily be Hilbert frames. Feichtinger and Grochenig [6] established a general theory applicable to a wide range of function spaces and group representations.

In [8], Grochenig explored atomic decompositions as a broader concept than frames. In this paper, we will build on this work by introducing the concept of approximative atomic decompositions for Banach spaces. This paper aims to provide a characterization of these decompositions. The paper has established that a Banach space E has an approximative atomic decomposition if it satisfies the bounded approximation property. Moreover, sufficient conditions for the existence of such decompositions in separable Banach spaces have been derived. As an application, it has been shown that if both Banach spaces E and F have the bounded approximation property, then the product space E × F with the $\ell 1$-norm also possesses this property.

2. Preliminaries

In the paper, E denotes an infinite-dimensional Banach space over the scalar field K (either real numbers or complex numbers). E^* and E^{**} refer to the first and second conjugate spaces of E, respectively. B(E) represents the Banach space comprising all continuous linear mappings on E, while L(E, E) denotes the Banach space of all linear mappings on E. Ed denotes a specific associated Banach space of scalar-valued sequences indexed by N (natural numbers).The notation [fn] denotes the closed linear span of the sequence { fn } in E^*, and [Þfn] represents the closed linear span of { fn } in the $\sigma(E^*, E)$-topology. A sequence { fn } ⊂ E^* is termed complete if its closed linear span [fn] equals E^*, indicating that it densely covers E^*. It is considered total if the set of elements x in E such that fn(x) = 0 for all n in N is solely the zero vector in E, meaning { fn } provides a full description of E∗ without leaving any significant gaps.

Definition 2.1 from reference [6] states: In a Banach space E with an associated Banach space Ed of scalar-valued sequences indexed by N, a pair ({ fn }, {xn }) is termed an atomic decomposition for E with respect to Ed if

(a) $\{f_n(x)\} \in E_d$, for all $x \in E$

(b) there exist constants A, B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad \text{for all } x \in E$$

(c) $x = \sum_{n=1}^{\infty} f_n(x)x_n$, for all $x \in E$.

The positive constants A, B are called atomic bounds for the atomic decomposition ({ f_n }, { x_n }).

Definition 2.2 from reference [12] defines the bounded approximation property for a Banach space E as follows: E is said to have the bounded approximation property if there exists a constant λ ≥ 1 such that the identity operator IE : E → E can be uniformly approximated on every compact subset of E by finite-rank linear operators with norm ≤ λ. Formally, for every compact subset Q ⊂ E and for every ε > 0, there exists an endomorphism u = uQ,ε ∈ L(E, E) (linear mappings from E to E) of finite rank, satisfying ‖u‖ ≤ λ, such that ‖u(x) − x‖ < ε for all x ∈ Q.

The result referenced in this paper is presented in the form of a lemma.

Lemma 2.3, as stated in references [11, 12], asserts that if E is a Banach space and { fn } ⊂ E∗ is a total sequence over E, then E is linearly isometric to the associated Banach space Ed = { { fn (x)} : x ∈ E }, where the norm in Ed is defined as ‖{ fn (x)}‖Ed = ‖x‖E for all x ∈ E. This means that E and Ed are essentially the same in terms of their linear structure and norm, despite being expressed in different forms (E via its elements and Ed via sequences indexed by { fn }).

3. Main Results

Definition 3.1 defines an approximative atomic decomposition for a Banach space E with respect to an associated Banach space Ed of scalar-valued sequences indexed by N. Specifically, given {xn} ⊂ E and {hn,i} for i = 1, 2, ..., mn (where mn is an increasing sequence of positive integers) ⊂ E∗, the pair ({hn,i}, {xn}) is termed an approximative atomic decomposition for E with respect to Ed if

(a) $\{h_{n,i}(x)\}_{\substack{i=1,2,\dots,m_n \\ n\in\mathbb{N}}} \in E_d$, for all $x \in E$

(b) there exist constants A, B with $0 < A \le B < \infty$ such that

$$A\|x\|_E \le \|\{h_{n,i}(x)\}_{\substack{i=1,2,\dots,m_n \\ n\in\mathbb{N}}}\|_{E_d} \le B\|x\|_E, \quad \text{for all } x \in E$$

(c) $x = \lim_{n\to\infty} \sum_{i=1}^{m_n} h_{n,i}(x)x_i$, for all $x \in E$.

Indeed, it can be observed that if ({ fn }, {xn}) is an atomic decomposition for E with respect to Ed, where Ed is an associated Banach space of scalar-valued sequences indexed by N, then for hn,i = fn, i = 1, 2, ..., n; n ∈ N, ({hn,i}, {xn}) forms an approximative atomic decomposition for E with respect to some associated Banach space Ed0. This relationship holds because

$$\lim_{n\to\infty} \sum_{i=1}^{n} h_{n,i}(x)x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f_i(x)x_i = x.$$

It can be observed that if hn,i(x) = 0 for all i = 1, 2, ..., n and n ∈ N, then by equation (3.1), x must be zero. Therefore, according to Lemma 2.3, there exists an associated Banach space Ed0 = {{hn,i(x)} : i=1,...,n, n∈N : x ∈ E} with the norm defined as ‖{hn,i(x)} : i=1,2,...,n, n∈N ‖Ed0 = ‖x‖E, where x belongs to E. In this space Ed0, ({hn,i} : i=1,2,...,n, n∈N, {xn}) constitutes an approximative atomic decomposition for E.

However, it's important to note that ({hn,i} : i=1,2,...,n, n∈N, {xn}) in itself may not necessarily be an atomic decomposition for E with respect to Ed0.

Based on the above discussion, we can make the following observations and queries:

(I) If E possesses an atomic decomposition with respect to Ed, then E also possesses an approximative atomic decomposition with respect to some Ed0.

(II) The relationship between Ed and Ed0 needs further investigation and remains an open question.

(III) The converse of observation (I) is affirmed (Remark 3.4).

Regarding the existence of approximative atomic decompositions, we provide the following characterization:

Theorem 3.2 states that a Banach space E has an approximative atomic decomposition if and only if there exists a sequence {vn} ⊂ B(E) consisting of non-zero endomorphisms of finite rank such that

$$t\ x = \sum_{i=1}^{\infty} v_i(x),\ x \in E \text{ and } \sup \left\| \sum_{i=1}^{n} v_i \right\| \le \lambda, \text{ for some } \lambda > 0.$$

Proof. Suppose {xn} ⊂ E and {hn,i} ⊂ E∗ are sequences such that ({hn,i} : i=1,2,...,mn, n∈N, {xn}) forms an approximative atomic decomposition for E with respect to an associated Banach space Ed, where {mn} is an increasing sequence of positive integers. Define

$$u_n(x) = \sum_{i=1}^{m_n} h_{n,i}(x)x_i, \quad x \in E, n \in \mathbb{N}.$$

Then, for each $n \in \mathbb{N}$, u_n is a well defined continuous linear mapping on E with $\dim u_n(E) < \infty$ such that $\lim_{n\to\infty} u_n(x) = x$, $x \in E$. Also, by using principle of uniform boundedness, $\sup_{1\le n<\infty} \|u_n\| < \infty$. Without any loss of generality, we may assume that $u_1 \neq 0$ and $u_n \neq u_{n+1}$, for all $n \in \mathbb{N}$. Define

$$v_1 = u_1, \quad v_{2n} = v_{2n+1} = \frac{1}{2}(u_{n+1} - u_n), \quad n \in \mathbb{N}.$$

Then $\{v_n\}$ is a sequence of non-zero endomorphisms of finite rank in $B(E)$ such that

$$\begin{aligned}\sum_{i=1}^{n} v_i(x) &= u_1(x) + \frac{1}{2}\{(u_2(x) - u_1(x)) + (u_2(x) - u_1(x))\} \\ &\quad + \frac{1}{2}\{(u_3(x) - u_2(x)) + (u_3(x) - u_2(x))\} + \cdots \\ &= u_n(x), \quad x \in E.\end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} v_i(x) = \lim_{n\to\infty} u_n(x) = x, \quad x \in E.$$

Also

$$\sup_{1\le n<\infty} \left\| \sum_{i=1}^{n} v_i \right\| = \sup_{1\le n<\infty} \|u_n\| < \infty.$$

Conversely, taking $u_n = \sum_{i=1}^{n} v_i$, $n \in \mathbb{N}$, we have $\lim_{n\to\infty} u_n(x) = x$, $x \in E$. Since for each $n \in \mathbb{N}$, $u_n(E)$ is finite dimensional, there exist a sequence $\{y_{n,i}\}_{i=m_{n-1}+1}^{m_n}$ in E and a total sequence $\{g_{n,i}\}_{i=m_{n-1}+1}^{m_n}$ in E^* such that

$$u_n(x) = \sum_{i=m_{n-1}+1}^{m_n} g_{n,i}(x) y_{n,i}, \quad x \in E, \quad n \in \mathbb{N},$$

where $\{m_n\}$ is an increasing sequence of positive integers with $m_0 = 0$. Define $\{x_n\} \subset E$ and $\{h_{n,i}\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}} \subset E^*$ by

$$x_i = y_{n,i}, \quad i = m_{n-1}+1, \ldots, m_n, \; n \in \mathbb{N} \quad \text{and}$$

$$h_{n,i} = \begin{cases} 0, & \text{if } i = 1, 2, \ldots, m_{n-1} \\ g_{n,i} & \text{if } i = m_{n-1}+1, \ldots m_n. \end{cases}$$

Then, for each $x \in E$, $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \sum_{i=1}^{m_n} h_{n,i}(x) x_i = \lim_{n\to\infty} u_n(x) = x. \tag{3.2}$$

Let $x \in E$ be such that $h_{n,i}(x) = 0$, for all $i = 1, \ldots, m_n$ and $n \in \mathbb{N}$, then by (3.2), $x = 0$. Therefore, by Lemma 2.3, there exists an associated Banach space $E_d = \{\{h_{n,i}(x)\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}} : x \in E\}$ with norm given by $\|\{h_{n,i}(x)\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}}\|_{E_d} = \|x\|_E$, $x \in E$.

Hence, $(\{h_{n,i}\}_{\substack{i=1,\ldots,m_n \\ n\in\mathbb{N}}}, \{x_n\})$ is an approximative atomic decomposition for E with respect to E_d. □

Note: For the converse part of Theorem 3.2, we do not need to assume

$$\sup_{1\le n<\infty}\left\|\sum_{i=1}^{n} v_n\right\| < \infty.$$

Let E = ℓ ∞. For each n ∈ N, define vn : E → E by vn (x) = ξn en , x = {ξn } ∞ i=1 ∈ ℓ ∞ Then ‖vn (x)‖ = ‖ξn en‖ = |ξn | ≤ ‖x‖, n ∈ N. Therefore vn ∈ B(E), n ∈ N. Further, note that dim vn (E) = dim(span{en }) = 1 and sup ∑n i=1 vi ≤ 1. But E is nonseparable. Hence E does not have an approximative atomic decomposition. In view of the above discussion, we have the following problem

Problem. Does a Banach space E has an approximative atomic decomposition (or atomic decomposition) if there exists a sequence {vn } ⊂ B(E) such that x =∑$^{\infty}$ i=1 vi (x), x ∈ E?

The following result describes the relationship between approximative atomic decompositions and the bounded approximation property.

Theorem 3.3. A Banach space E has approximative atomic decomposition if and only if E has bounded approximation property.

Proof: Let ({hn,i} : i=1,2,...,mn, n∈N, {xn}) be an approximative atomic decomposition for E with respect to Ed, where {mn} is an increasing sequence of positive integers. Therefore, following the approach in Theorem 3.2, there exists

linear mappings {un } with dim un (E) < ∞ such that limn→∞ un (x) = x, x ∈ E and sup 1≤n< ∞. Let Q be any compact subset of E. Let ε > 0 be arbitrary and { yi } r i=1 be a finite ε 2(1+λ) -net for Q, where λ = sup 1≤n< ∞. Then, for any x ∈ Q, there exists some j such that

$$\|x - y_j\| < \frac{\epsilon}{2(1+\lambda)}, \quad 1 \le j \le r.$$

Since for each $i = 1, 2, \ldots r$, $y_i \in E$, there exists a positive integer n_i such that

$$\|y_i - u_n(y_i)\| < \frac{\epsilon}{2}, \text{ for all } n \ge n_i, \text{ and } i = 1, 2, \ldots, r.$$

Thus

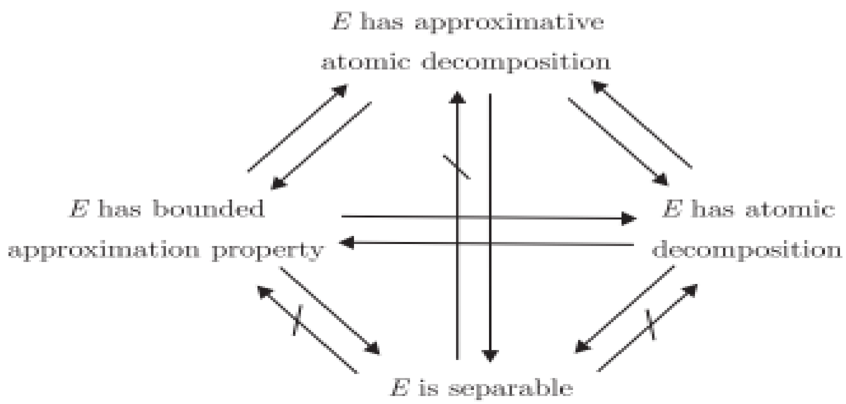
$$\begin{aligned}\|x - u_n(x)\| &\le \|x - y_j\| + \|y_j - u_n(y_j)\| + \|u_n(y_j) - u_n(x)\| \\ &< \epsilon, \quad \text{for all } n \ge n_i.\end{aligned}$$

Since $x \in Q$ and $\epsilon > 0$ is an arbitrary, we have

$$\lim_{n\to\infty} \sup_{x\in Q} \|x - u_n(x)\| = 0.$$

Thus, E possesses the bounded approximation property. The converse part of the theorem can be derived from Theorem 2.10 in reference [1] and Observation (I).

Remark 3.4: Based on Theorem 3.3 and Theorem 2.10 in [1], we can conclude that a Banach space E possesses an approximative atomic decomposition if and only if it has an atomic decomposition.It can be observed that if E has an approximative atomic decomposition, then E must be separable. However, according to Remark 3.4 and the example provided in [4], the converse statement does not necessarily hold true. Below is a diagram illustrating the relationships among these various concepts.



Given that a separable Banach space does not always guarantee an approximative atomic decomposition, it is pertinent to inquire about conditions under which such a space does possess this property. The following results provide sufficient conditions for a separable Banach space to have an approximative atomic decomposition.

Theorem 3.5. A separable Banach space E has an approximative atomic decomposition if there exists a constant λ ≥ 1 such that for every finite dimensional subspace G of E and every δ > 0 there exists a finite rank endomorphism u ∈ B(E) satisfying

$$\|u\| \leq \lambda \ \ and \ \ \|u(x) - x\| < \delta\|x\|, \quad x \in G.$$

Proof. Let { yn } be a sequence in E such that [yn] = E. Then, by hypotheses, for each n ∈ N there exists a finite rank endomorphism un ∈ B(E) such that

$$\|u_n\| \leq \lambda, \quad n \in \mathbb{N} \quad \text{and}$$

$$\|u_n(x) - x\| < \frac{1}{n}\|x\|, \quad x \in G_n = [y_1, y_2, \ldots, y_n], \ \ n \in \mathbb{N}.$$

Since Gn ⊂ Gn+1 , n ∈ N and ‖un‖ < λ, n ∈ N, it follows that limn→∞ un (x) = x, x ∈ E. Also sup 1≤n< ∞.

Hence, proceeding as in Theorem 3.2, E has an approximative atomic decomposition.

In reference [10], it has been demonstrated that if two Banach spaces possess Banach frames, then their product space also possesses a Banach frame. Similarly, the following result pertains to approximative atomic decompositions.

Finally, as an application of approximative atomic decompositions, we have the following corollary.

Corollary 3.7. If E and F are Banach spaces having bounded approximation property, then the product space E × F also has bounded approximation property.

Proof. Follows in view of Theorem 3.3.

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