JETIR.ORG ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue
JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)
An International Scholarly Open Access, Peer-reviewed, Refereed Journal

SOME SETS OF INTEGERS AND THEIR TRANSFORMATIONS

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Abstract

In this Paper integers have been divided modulo 5, **[1]** and classes A_1, A_2, A_3, A_4, A_5 have been defined. We consider sums and multiplication of these sets and get a ring (R,+,•). We consider some linear transformations on R and show that these transformations form a group as well as a ring, named K. This K is also a field **[2]**.

Key words:- Group, Ring, Field, Modules, linear transformations

1. <u>Definition of addition and its Consequences:-</u>

Let us divide integers modulo 5. Thus all numbers of the form 5K+1.K bring any integer will be denoted by $\overline{1}$ or A_1. Numbers of form 5K+2 will be denoted by $\overline{2}$ or A_2 and so on. Thus we get $\overline{A}_3$ and $\overline{A}_4$. Finally numbers of form 5K will be denoted by $\overline{5}$ of $\overline{0}$ by A_5.

Then clearly,

$A_1+A_2= \overline{1}+\overline{2}= \overline{3}= A_3$, $A_1+A_3= \overline{1}+\overline{3}=\overline{4} = A_4$, $A_1+A_4= \overline{1}+\overline{4} = \overline{5} = A_5$,

$A_1+A_5= \overline{1}+\overline{5} = \overline{1} = A_1$,

$A_2 + A_2= \overline{2} +\overline{2} = \overline{4} = A_4$, etc.

Thus in general,

$A_i + A_j = A_{i+j}$ (mod 5)

By These calculations, we can construct the following table,

+	A_1	A_2	A_3	A_4	A_5
A_1	A_2	A_3	A_4	A_5	A_1
A_2	A_3	A_4	A_5	A_1	A_2

A_3	A_4	A_5	A_1	A_2	A_3
A_4	A_5	A_1	A_2	A_3	A_4
A_5	A_1	A_2	A_3	A_4	A_5

In above table, A_5 behaves like an identity element for operation '+'. Also since $A_1+A_4= A_5$, we can say that additive inverse of A_1 is A_4 i.e.

$-A_1= A_4$

Similarly $-A_2= A_3$, $-A_3= A_2$, $-A_4= A_1$ and $-A_5= A_5$

As we deal with real numbers, associative law can be easily checked e.g. $(A_1+A_2) +A_3= A_3+A_3= A_1$,

$A_1+ (A_2+A_3) = A_1+A_5=A_1$ Showing $(A_1+A_2)+A_3= A_1+(A_2+A_3)$

Hence $\{A_1, A_2, A_3, A_4, A_5\}$ with operation '+' is an abelian group with identity A_5.

2. Definition of multiplication and its consequences:-

Now we define multiplication, e.g.

$A_1A_1= (\bar{1})(\bar{1})= \bar{1}= A_1$,

$A_1A_2= (\bar{1})(\bar{2})= \bar{2}= A_2$

Similarly $A_1A_3= A_3$, $A_1A_4= A_4$, and $A_1A_5= A_5$, $A_2A_2= (\bar{2})(\bar{2})= \bar{4}= A_4$,

$A_2A_3= (\bar{2})(\bar{3})= \bar{1}= A_1$ etc.

So in general,

$A_iA_j= A_{ij\ (\mathrm{mod}\ 5)}$

With these calculations we get the following table,

•	A_1	A_2	A_3	A_4	A_5
A_1	A_1	A_2	A_3	A_4	A_5
A_2	A_2	A_4	A_1	A_3	A_5
A_3	A_3	A_1	A_4	A_2	A_5
A_4	A_4	A_3	A_2	A_1	A_5
A_5	A_5	A_5	A_5	A_5	A_5

Here with operation '•' A_1 behaves as identity element.Since $A_1A_1= A_1$,

$A_2A_3= A_1, A_4A_4= A_1$

We can write

$A_1^{-1}= A_1, A_2^{-1}= A_3$ $A_3^{-1}= A_2$ $A_4^{-1}= A_4$

We can check associative laws, e.g. $(A_1A_2)A_3= A_2A_3= A_1$, $A_1(A_2A_3)= A_1A_1= A_1$ $(A_2A_3)A_4= A_1A_4= A_4$

$A_2(A_3A_4)= A_2A_2= A_4$ etc

Hence $\{A_1, A_2, A_3, A_4\}$ together with '•' forms an abelian group of which

$\{A_1, A_4\}$ is a subgroup.

Now we check distributive laws, e.g.

$A_1(A_1+A_2)= A_1A_3= A_3$, $A_1A_1+A_1A_2= A_1+A_2= A_3$ $A_2(A_1+A_2)= A_2A_3= A_1$, $A_2A_1+A_2A_2= A_2+A_4= A_1$ etc

Hence considering previous result, we see that

$(\{A_1, A_2, A_3, A_4, A_5\},+,\bullet)$= R(say) is an abelian ring with multiplicative identityA_1 ($\neq A_5$) and A_5 being identity for '+'. Also R has no zero divisors, so it is an integral domain. We know that every finite integral domain is a field. Hence it is a field also.

Now we consider powers of the elements of R.

$A_1^2= A_1$, $A_2^2= A_4$, $A_3^2= A_4$, $A_4^2= A_1$, $A_5^2= A_5$ $A_1^3= A_1^2A_1= A_1A_1= A_1$, $A_2^3= A_2^2A_2= A_4A_2= A_3$

$A_3^3= A_3^2A_3= A_4A_3= A_2$, $A_4^3= A_4^2A_4= A_1A_4= A_4$ $A_5^3= A_5^2A_5= A_5A_5= A_5$

$A_1^4= A_1$, $A_2^4= (A_2^2)^2= A_4^2= A_1$, $A_3^4= A_3^3A_3= A_2A_3= A_1$

$A_4^4= A_4^3A_4= A_4A_4= A_1$, $A_5^4= A_5$ $A_1^5= A_1$, $A_2^5= A_2^4A_2= A_1A_2= A_2$, $A_3^5= A_3^4A_3= A_1A_3= A_3$

$A_4^5= A_4^4A_4= A_1A_4= A_4$

$A_5^5= A_5^4A_5= A_5A_5= A_5$

So in general , $A_i^5= A_i$ (i= 1,2,…,5)Also $A_i^6= A_i^5A_i= A_iA_i= A_i^2$ etc

So we conclude that $A_i^n= A_i^m$ where $n\equiv m \pmod 4$We get table for powers of A_i

Index	2	3	4	5
A_1	A_1	A_1	A_1	A_1
A_2	A_4	A_3	A_1	A_2
A_3	A_4	A_2	A_1	A_3
A_4	A_1	A_4	A_1	A_4
A_5	A_5	A_5	A_5	A_5

Next we consider integral multiples of A_i's.

$2A_1= A_1+A_1= A_2$,

$2A_2= A_2+A_2= A_4$,

$2A_3= A_3+A_3= A_1\equiv A_6$

$2A_4= A_4+A_4= A_3\equiv A_8$,

$2A_5= A_5\equiv A_{10}$

$3A_1= 2A_1+A_1= A_2+A_1\equiv A_3$,

$3A_2= A_4+A_2\equiv A_1$, etc

So in general

$mA_i= A_j$ where $j\equiv mi \pmod 5$

we can make a table regarding integral multiples as following,

•	A_1	A_2	A_3	A_4	A_5
2	A_2	A_4	A_1	A_3	A_5
3	A_3	A_1	A_4	A_2	A_5
4	A_4	A_3	A_2	A_1	A_5
5	A_5	A_5	A_5	A_5	A_5

We show that

$(i+j)A_k= iA_k+jA_k$, e.g $(1+3)A_1= 4A_1= A_4$, $A_1+3A_1= A_1+A_3= A_4$ etc

We have proprety such as

$a(A_i+A_j)= aA_i+aA_j$, e.g.$2(A_3+A_4)= 2A_2= A_4$

$2A_3+2A_4= A_1+A_3= A_4$ etc

We have property such as

$(ab)A_i= a(bA_i)$, e.g.

$(2\times3)A_1= 6A_1= A_1$, $2(3A_1)= 2A_3= A_1$ etc.

Also $1.A_i= A_i$

So the system $(\{A_1, A_2, A_3, A_4, A_5\}, +,\bullet)$ forms a system analogous to a vector space over the set of integers.

Let us call it V.

3. Some linear transformations on R:-

Define $S:R\longrightarrow R$ by

$S(A_i)= A_iA_2$ (i.e. multiplying by 2, 7, ….etc)Then $S(A_1)= A_1A_2= A_2$

$S(A_2)= A_2A_2= A_4$ $S(A_3)= A_3A_2= A_1$ $S(A_4)= A_4A_2= A_3$ $S(A_5)= A_5A_2= A_5$

We show that S is linear, i.e.

$S(A_i+Aj)= S(Ai)+S(Aj)$ and $S(\alpha A_i)= \alpha s(A_i)$ Now $S(A_i+A_j)= (A_i+A_j)A_2= A_iA_2+A_jA_2=$

$S(A_i)+S(A_j)$

And $S(\alpha A_i)= (\alpha A_i)A_2= \alpha(A_iA_2)= \alpha\ S(A_i)$Where α varies from 1 to 5.

Also

$S^2(A_i)= S(S(A_i))= S(A_iA_2)= (A_iA_2)A_2 = A_i(A_2A_2)= A_iA_4$

Hence

$S^3(A_i)= S(S^2(A_i))= S(A_iA_4)= (A_iA_4)A_2 = A_i(A_4A_2)= A_iA_3$

Then

$S^4(A_i)= S(S^3(A_i))= S(A_iA_3)= (A_iA_3)A_2 = A_i(A_3A_2)= A_iA_1= A_i$

So we conclude that

$S^4= I$, identity transformation on RTherefore $S^5= S$

Next we define $T:R\longrightarrow R$ by

$T(A_i)= A_iA_3$ (or multiplication by 3, 8,…)

Then

$T(A_1)= A_1A_3= A_3$, $T(A_2)= A_2A_3= A_1T(A_3)= A_3A_3= A_4$, $T(A_4)= A_4A_3= A_2$

And $T(A_5)= A_5A_3= A_5$

As in case of S, we can easily see that T is linear.

Also

$$T^2(A_i)= T(T(A_i))= T(A_iA_3)= (A_iA_3)A_3$$

$$= A_i(A_3A_3)= A_iA_4$$

$$T^3(A_i)= T(T^3(A_i))= T(A_iA_4)= (A_iA_4)A_3$$

$$= A_i(A_4A_3)= A_iA_2$$

$$T^4(A_i)= T(T^3(A_i))= T(A_iA_2)= (A_iA_2)A_3$$

$= A_i(A_2A_3)= A_iA_1= A_i$

Hence $T^4= I$ and $T^5= T$ Next we define $U:R\longrightarrow R$ by

$U(A_i)= A_iA_4$ (or multiplication by 4, 9,…)

Then

$U(A_1)= A_1A_4= A_4U(A_2)= A_2A_4= A_3U(A_3)= A_3A_4= A_2U(A_4)= A_4A_4= A_1$

$U(A_5)= A_5A_4= A_5$

We can easily see that U is linear

Also $U^2(A_i)= U(U(A_i)) = U(A_iA_4)= (A_iA_4)A_4A_i(A_4A_4) = A_iA_1= A_i$

Hence $U^2=I$

$\Longrightarrow$ $U^3= U$ i.e. U is a trijection.

$\Longrightarrow$ $U^4= U^2= I$

$\Longrightarrow$ $U^5= U$

Since $T^2(A_i)= A_iA_4$, hence $T^2= U$ ....(3.1)

We show that T and S commute, since

$(TS)A_i= T((S(A_i))= T(A_iA_2)= (A_iA_2)A_3= A_i(A_2A_3)= A_iA_1= A_i$ And $(ST)A_i= S((T(A_i))= S(A_iA_3)= (A_iA_3)A_2= A_i(A_3A_2)= A_iA_1= A_i$ So $TS = ST = I$

We show that T commutes with U, since

$(UT)(A_i)= U(T(A_i))= U(A_iA_3)= (A_iA_3)A_4= A_i(A_3A_4)= A_iA_2= S(A_i)$ $(TU)(A_i)= T(U(A_i))= T(A_iA_4)= (A_iA_4)A_3= A_i(A_4A_3)= A_iA_2= S(A_i)$

Hence $TU= UT= S$

Now we show that U commutes with S, since$US= U(UT)= U^2T = IT= T$

And $SU= (TU)U= TU^2= TI= T$

Hence $US= SU= T$

Also since $S^2(A_i)= A_iA_4= U(A_i)$, hence $S^2= U$ .... (3.2)

Then

$S^3 = SS^2 = SU = T$, $S^4 = (S^2)^2 = U^2 = I$

We have due to (3.1) and (3.2)$S^2 = T^2 = U$

So combining above results,

TS= ST= I, TU= UT= S, US= SU= T

We can verify associative laws, e.g.

$T(US) = T.T = T^2 = U$, $(TU)S = SS = S^2 = U$ etc

Hence we make a table with I, S, T, U and multiplication as binary operation.

•	I	S	T	U
I	I	S	T	U
S	S	U	I	T
T	T	I	U	S
U	U	T	S	I

Thus set (I, S, T, U) form an abelian group with identity I where,
$I^{-1} = I$, $S^{-1} = T$, $T^{-1} = S$ and $U^{-1} = U$
Associative law for multiplication has been also checked.

Now we consider sum of operators.

$(S+S)(A_i) = S(A_i) + S(A_i)$

$= A_iA_2 + A_iA_2$

$= A_i(A_2 + A_2)$

$= A_iA_4$

$= U(A_i)$

Hence S+S= U

$(T+T)(A_i) = T(A_i) + T(A_i)$

$= A_iA_3+A_iA_3$

$= A_i(A_3+A_3)$

$= A_iA_1$

$= A_i$

$= I(A_i)$

So $T+T= I$

$(U+U)(A_i)= U(A_i)+U(A_i)$

$= A_iA_4+A_iA_4$

$= A_i(A_4+A_4)$

$= A_iA_3$

$= T(A_i)$

Therefore $U+U= T$

$(I+I)(A_i)= I(A_i)+I(A_i)= A_i+A_i$

So $(I+I)(A_1)= I(A_1)+I(A_1)= A_1+A_1= A_2= S(A_1)$ $(I+I)(A_2)= I(A_2)+I(A_2)= A_2+A_2= A_4= S(A_2)$, $(I+I)(A_3)= I(A_3)+I(A_3)= A_3+A_3= A_1= S(A_3)$ $(I+I)(A_4)= I(A_4)+I(A_4)= A_4+A_4= A_3= S(A_4)$ $(I+I)(A_5)= I(A_5)+I(A_5)= A_5+A_5= A_5= S(A_5)$

Hence $I+I= S$

$(S+T)A_i= S(A_i)+T(A_i)= A_iA_2+A_iA_3= A_i(A_2+A_3)= A_iA_5=A_5= 0(A_i)$

Where 0 is zero operator such that $0(A_i)= A_5$, Hence $S+T= 0$

$(S+U)(A_i)= S(A_i)+U(A_i)= A_iA_2+A_iA_4= A_i(A_2+A_4)= A_iA_1= A_i= I(A_i)$

Hence $S+U= I$

$(T+U)(A_i)= T(A_i)+U(A_i)= A_iA_3+A_iA_4= A_i(A_3+A_4)= A_iA_2= S(A_i)$

Hence $T+U= S$, Also

I+S= (S+U)+S= S+(U+S)= S+(S+U)= (S+S)+U= U+U= T, Similarly I+T= (S+U)+T= S+(U+T)= S+S= U

And I+U= (S+U)+U= S+(U+U)= S+T= 0

Hence when we consider {0, I, S, T, U} with '+', we obtain the following table.

+	0	I	S	T	U
0	0	I	S	T	U
I	I	S	T	U	0
S	S	T	U	0	I
T	T	U	0	I	S
U	U	0	I	S	T

Thus it is an abelian group with 0 as identity element.Also -I= U, -S= T, -T= S and – U=I

Distributive laws with '+' and '•' can be verified e.g.

T(U+T)= TS= I, TU+TT= S+U= I etc

Also, since ({I, S, T, U},•) forms an abelian group, does not have zero divisorsand distributive laws hold, hence the system,

K= ({0, I, S, T, U}, +,•) forms an abelian ring with identity I, zero 0 and no zerodivisors. Thus it forms an integral domain. Since every finite integral domain is a field, it is also a field.

Reference:-

[1] Simmons, G.F,

"*Introduction To Topology And Modern Analysis*" McGraw-Hill book Company, Inc., New York,(1963). PP.181-193

[2] Simmons, G.F,

"*Introduction To Topology And Modern Analysis",* McGraw-Hill book Company, Inc., New York, (1963), PP.200-206

[3] Penney, Richard C. :

"*Linear Algebra*" Second Edition, John Wiley & Sons, Inc.,

Hoboken, New Jersey

[4] Pinter Charles C:

"*A Book Of Abstract Algebra*" Second Edition Dover PublicationInc, Mineola, New York (2013)

[5] Pundir S.K. :

"*Functional Analysis*" CBS Publisher And Distributors India(2016)