EXACT SOLUTIONS FOR SPHERICALLY SYMMETRIC CHARGED FLUID DISTRIBUTION

By :

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Abstract :

The present paper provides exact solutions for charged fluid sphere by the method of quatrature. Here we have used a suitable form of metric potential and total charge Q(r). To fix the constants appearing in the solution, we have used boundary conditions where we have matched the solutions with exterior Reissner-Nordstrom metric.

Key Words :

Charge fluid, quadrature, metric, potential, perfect fluid.

1. INTRODUCTION

Various relativists have shown their keen interest towards the study of charged matter distribution. Charged fluid distribution in equilibrium has been investigated by a pretty number of researches like Bonnor [2,3,4], Majumdar [15], Papapetrou [19], De and Ray Chaudhari [7] and Ibrahim et. al. [12]. The problem of charged fluid spheres Under different conditions have been studied by different allthors like Efinger [8], Kyle and Martin, Florides [8, 9, 10], Chakravarti and De [6] and many others. The generalization of Nordstrom's solution corresponding to the external field of a radiating charged particle

has been give by Shah [22]. Exact solution for the static charged sphere has been obtained by Wilson [27]. Some other workers in this field are Bonnor and Wickramasuriya [5], Bailyn and Eimeral [1] Omote [18], Krori and Barua [14], Mehra [16], Singh et. al. [25], Nduka [17], Singh and Yadav [24], Koppar et. al. [13], Sah and Chandra [21], Purushottam and Yadav [20], Yadav et. al. [28, 29] and Singh and Kumar [23]. As a matter of fact exact solution of Einstein – Maxwell (EM) equations are difficult to find as they are non linar. Therefore it is necessary to specify the unknowns or to put extra relations between them to solve these equations.

The present chapter provides exact solutions for charged fluid sphere by the method of quadratures. Here we have used a specific choice of metric potential and total charge Q(r). To fix the constant appearing in the solution we have used boundary conditions where we have matched the solutions with exterior Reissner-Nordstrom metric.

2.   FIELD EQUAITONS

We consider static spherically symmetric metric given by

(2.1) $ds^2 = \upsilon^2 dt^2 - \lambda^2 dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$

For perfect fluid, the Einstein – Maxwell field equations are

(2.2) $G_{ij} = -8\pi\left(T_{ij} + E_{ij}\right)$

where T_{ij} is the energy momentum tensor given by

(2.3) $T_{ij} = (\rho + p)u_i u_j - pg_{ij}$

u_{ij} being four velocity and E_{ij} the energy momentum tensor of the electromagnetic field given by

(2.4) $E_{ij} = \dfrac{1}{4\pi}\left[g^{k\ell}F_{ik}F_{j\ell} - \dfrac{1}{4}g_{ij}F_{k\ell}F^{k\ell} \right]$

F_{ij} is electromagnetic field tensor. Also the electromagnetic field equations are given by

$$(2.5) \quad \left[(-g)^{1/2} F^{ij} \right]_{,j} = 4\pi J^i (-g)^{1/2}$$

and

$$(2.6) \quad F_{[ij;k]} = 0$$

where j^i is the four-current.

we have choosen natural units of that $C = G = 1$.

　　　Using comoving coordinates, we can write the field equations as

$$(2.7) \quad 8\pi p = \frac{1}{\lambda^2} \left[\frac{2\upsilon'}{r\upsilon} + \frac{1}{r^2} \right] - \frac{1}{r^2} + \frac{2Q^2}{r^4}$$

$$(2.8) \quad 8\pi p = \frac{1}{\lambda^2} \left[\frac{\upsilon}{r} - \frac{\upsilon'\lambda'}{\upsilon\lambda} + \frac{1}{r} \left(\frac{\upsilon'}{r} - \frac{\lambda'}{\lambda} \right) \right] + \frac{Q^2}{r^2}$$

$$(2.9) \quad 8\pi\rho = \frac{1}{\upsilon^2} \left(\frac{2\upsilon'}{r\upsilon} - \frac{1}{r^2} \right) + \frac{1}{r^2} + \frac{Q^2}{r^4}$$

Where a prime denotes differentiation w.r.t. r and Q is the total charge within the sphere of radius r given by

$$(2.10) \quad Q(r) = 4\pi \int_0^r J^4 x^2 \upsilon\lambda \, dx$$

The corresponding electric field is given by

$$(2.11) \quad F^{14} = \frac{Q(r)}{\lambda\upsilon r^2}$$

　　　From equation (2.7) and (2.8) eliminating p, we obtain

$$(2.12) \quad \frac{d}{dr} \left(\frac{1-\lambda^2}{\lambda^2 r^2} \right) + \frac{d}{dr} \left(\frac{\upsilon'}{\lambda^2 \upsilon r} \right) + \left(\frac{1}{\lambda^2 \upsilon^2} \right) \frac{d}{dr} \left(\frac{\upsilon\upsilon'}{r} \right) + \frac{4Q^2}{r^5} = 0$$

Taking $\dfrac{\upsilon'}{\upsilon r} = \mu(r)$ equation (2.12) can be cast into the form

(2.13) $\left(\dfrac{1}{r^2\lambda^2} + \dfrac{\mu}{\lambda^3}\right)d\lambda - \left(\dfrac{1}{\lambda^2}\right)d\mu - \left(\dfrac{\lambda^2 r}{\lambda^2} + \dfrac{Q^2}{r^5}\right)dr = 0$

3.   SOLUTIONS OF THE FIELD EQUAITONS

To solve equation (2.13) we write it in the form

(3.1) $\dfrac{d\lambda}{dr} = \dfrac{(1+Q^2)}{r^3\left[\dfrac{1}{r^2} + \mu\right]} + \left[\dfrac{-\dfrac{1}{r^3} + \mu^2 r + \dfrac{d\mu}{dr}}{\dfrac{1}{r^2} + \mu}\right]\lambda$

Letting $\lambda^2 = \dfrac{1}{\alpha}$, we can write (3.1) in the form

(3.2) $\dfrac{d\alpha}{dr} + 2\left[\dfrac{\dfrac{-1}{r^3} + \mu^2 r + \dfrac{d\mu}{dr}}{\dfrac{1}{r^2} + \mu}\right]\alpha = -\dfrac{2(1+Q^2)}{r^2\left[\dfrac{1}{r^2} + \mu\right]}$

Thus if $\mu(r)$ i.e. $\upsilon(r)$ and $Q(r)$ are known, we can find $\lambda(r)$ by using linear differential equation (3.2). A number of interesting cases arise

Case – I

We write

(3.3) $-\dfrac{-\dfrac{1}{r^3} + \mu^2 r + \dfrac{d\mu}{dr}}{\dfrac{1}{r^2} + \mu} = -\dfrac{1}{r}$

Then

(3.4a) $\mu(r) = \dfrac{1}{\mu_0 r + r^2}$

(3.4 b) $\upsilon^2\left(A_0 + A_1 r\right)^2$

Solving (3.2) we get the solution as

$$(3.5) \quad \alpha(r)\frac{1}{\lambda^2} = 1 + \lambda_0 r^2 - \left(\frac{2}{\mu_0}\right) + \left(\frac{4r^2}{\mu_0^2}\right)\log[(\mu_0 + 2r)/r] - \frac{2\zeta}{(n-2)}r^n$$

$$+2\zeta\, r^2 \int \frac{r^{n-2}}{(\mu_0 + 2r)}dr$$

where we have choosen

$$(3.6) \quad Q(r) = \zeta r^n$$

μ_0, λ_0 and ζ are constants.

Now putting the values of λ, υ and Q in equation

(2.7) we immediately get

$$(2.3.7) \quad p = \frac{1}{8\pi}\left[2\zeta r^{n-4} - \frac{1}{r^2} + \frac{(A_0 + 3A_1 r)}{r^2 (A_0 + A_1 r)} \right]\left\{ 1 + \lambda_0 r^2 \right.$$

$$-(2) - \left(\frac{2}{\mu_0}\right)(\mu_0)r + \left(\frac{4r^2}{\mu_0^2}\right)\log\left[\left(\frac{\mu_0 + 2r}{r}\right)\right]$$

$$-\frac{2\zeta}{(n-2)}r^n + 2\zeta r^2 \int \frac{r^{n-2}}{(\mu_0 + 2r)}dr \left.\vphantom{\frac{}{}}\right\}\right]$$

In the same way equation (2.9) provides

$$(3.8) \quad \rho = \frac{1}{8\pi}\left[\zeta^2 r^{2n-4} + \frac{1}{\mu_0 r} + \frac{2(n+1)\zeta r^{n-2}}{(n-2)} \right.$$

$$-\frac{12}{\mu_0^2}\log\left(\frac{\mu_0 + 2r}{r}\right) + \frac{4}{\mu_0(\mu_0 + 2r)} - \frac{2\zeta r^{n-1}}{(\mu_0 + 2r)}$$

$$6\zeta \int \frac{r^{n-2}}{(\mu_0 + 2r)}dr - 3\lambda_0 \left.\vphantom{\frac{}{}}\right]$$

Next in the second possibility, we solve $\mu(r)$ if $\lambda(r)$ is known. We can reduce equation

(2.13) to the form

$$(3.9) \quad \frac{d\mu}{dr} = \left(\frac{1}{\lambda r^2} \frac{d\lambda}{dr} - \frac{\lambda^2 - 1}{r^3} \right) + \left(\frac{1}{\lambda} \frac{d\lambda}{dr} \right) \mu - \mu^2 r + \frac{\lambda^2 Q^2}{r^5}$$

For known $Q(r)$ given by (3.6), the equation (3.9) is a Riccati equation for $\mu(r)$ and in general very difficult to solve. So we try for some cases of physical interest.

Case – II

Here we choose $\lambda = \beta\mu(r)$

were β is a constant then we find the solution as

$$(3.10) \quad \mu = \left[\beta^2 + k_1 r^2 - r^4 + \frac{\beta^2 \zeta^2 r^{2n-2}}{(n-2)} \right]^{-1/2}$$

Where $n \neq 2$,

Subcase (i) $n = 1$

$$(3.11) \quad \log \upsilon = \frac{1}{2} \sin^{-1} \left[\frac{(-k_1 + 2r^2}{\left\{ k^2 + 4\beta^2(1 - \zeta^2) \right\}^{12}} \right] + k_2$$

$$(3.12) \quad \lambda = \beta \left[\beta^2 + k_1 r^2 - r^4 - \beta^2 \zeta^2 \right]^{-1/2}$$

By use of equation (3.11) and (3.12) the equations (2.7) and (2.9) give, respectively

$$(3.13) \quad p = \frac{1}{\beta \pi p^2} \left[2\left(\beta^2 + k_1 r^2 - r^4 - \beta^2 \zeta^2 \right)^1 2 + k_1 + r^2 \right] + \frac{\zeta^2}{8\pi r^2}$$

$$(3.14) \quad \rho = \frac{1}{8\pi} \beta^2 [5r^2 - 3k_1] + \frac{\zeta^2}{4\pi r^2}$$

Subcase (ii) n = 3

In this case, we find

$$(3.15) \quad \log \upsilon = \frac{1}{2(1 - \zeta^2 \beta^2)^{1/2}} \sin^{-1} \left[\frac{2r^2(1 - \zeta^2 \beta^2)}{k^2 + 4\beta^2(1 - \zeta^2)\beta^2} \right] + \mu$$

$$(3.16) \quad \lambda = \beta \left[\beta^2 + k_1 r^2 - (1 - \zeta^2 \beta^2)r^4 \right]^{-1/2}$$

Using equations (3.15) and (3.16) in (2.7) we get

$$(3.17) \quad p = \frac{1}{8\pi\beta^2}\left[2\left\{\beta^2 + k_1r^2 - (1 - \beta^2\zeta^2r^4)^{1/2} + k_1 - r^2\right\}\right] + \frac{3\zeta^2r^2}{8\pi}$$

In a similar way from equation (2.9), we find

$$(3.18) \quad \rho = \frac{1}{8\pi\beta^2}[5r^2 - 3k_1] - \frac{\zeta^2r^2}{2\pi}$$

4.   BOUNDARY CONDITIONS

Here we match the different obtained interior solutions with exterior Reissour Nordstrom metric. The line element for $r > r_0$ is given by Reissner-Nordstrom metric given by (r_0 is boundary)

$$(4.1) \quad ds^2 = Ydt^2 - Y^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where $\gamma = \left(1 - \frac{2M}{r} - \frac{Q_0}{r^2}\right), Q_0 = Q(r_0)$ and m is the total mass of sphere given by

$$M = 4\pi\int_0^{r_0}\rho(r)r^2dr^2$$

we use the following boundary conditions.

1.   The metric potentials g_{44} and g_{11} are continuous across the boundary ($r = r_0$) of the fluid sphere.

2.   Derivative of g_{44} i.e. $\frac{d\upsilon}{dr}$ is continuous across the boundary ($r = r_0$) of the fluid sphere.

We use these boundary conditions to fix the constants appearing in the solution.

Case – I

$$(4.2) \quad \left(A_0 + A_1 r_0\right)^2 = \left(1 - \frac{2m}{r_0} + \frac{Q_0^2}{r_0^2}\right)$$

$$(4.3) \quad \lambda_0 r_0^2 - 2r_0^3 \frac{2\zeta r_0^{n+3}}{(n+1)} + 2r_0 \int_0^{r_0} \frac{r + \zeta r^{n+1}}{\mu_0 + 2r} dr = \left(1 - \frac{2M}{r_0} + \frac{Q_0^2}{r_0^2}\right)$$

$$(4.4) \quad A_1(A_0 + A_1 r_0) = \frac{M}{r_0^2} + \frac{Q_0^2}{r_0^3}$$

Subcase (i) n = 1

In this case the boundary conditions are given by

$$(4.5) \quad \left(A_0 + A_1 r_0\right)^2 = \left[1 - \frac{2M}{r_0} + \frac{Q^2}{r_0^2}\right]$$

$$(4.6) \quad \left[\lambda_0 + 1 - \frac{\zeta\mu_0}{2}\right] r_0^2 - \left(2 - \frac{\zeta}{2}\right) r_0^3 - K r_0^4$$

$$+ \frac{1}{4}(\zeta\mu_0^2 - 2\mu_0) r_0 \log(\mu_0 + 2r) = \left(1 - \frac{2M}{r_0} + \frac{Q_0^2}{r_0^2}\right)$$

$$(4.7) \quad A_1(A_0 + A_1 r_0) = \frac{M}{r_0^2} - \frac{Q_0^2}{r_0^3}$$

Solving (4.5), (4.6) and (4.7) we get the constant A_0, A_1 λ_0 and μ_0 in terms of M, Q_0 and r_0 where r_0 is obtained by equating p to zero.

Subcase (ii), n = 3

$$(4.8) \quad \left(A_0 + A_1 r_0\right)^2 = \left(1 - \frac{2M}{r_0} + \zeta^2 r_0^4\right)$$

$$(4.9) \quad (\zeta\mu_0^3 - 8)\frac{\mu_0 r_0}{16} \log(\mu_0 + 2r_0) + \left(\lambda_0 + 1 - \frac{\zeta\mu_0^3}{8}\right) r_0^2$$

$$+\left(\frac{\zeta\mu_0^3}{8}-2\right)r_0^3\frac{\zeta\mu_0+r_0^4}{6}+\frac{\zeta r_0^5}{4}=\left[1-\frac{2M}{r_0}+\zeta^2 r_0^2\right]$$

(4.10) $\quad 2A_1(A_0+A_1 r_0)=\dfrac{2M}{r_0^2}+4\zeta^2 r_0^3$

From equation (4.8) and (4.10), we have

(4.11) $\quad \dfrac{A_0}{A_1 r_0^2}+\dfrac{1}{r_0}=\dfrac{1+\zeta^2}{M}-\dfrac{2}{r_0}$

which shows that

(4.12) $\quad \dfrac{A_0}{A_1 r_0^2}+\dfrac{3}{r_0}>0$

or

(4.13) $\quad A_0>-3A_1 r_0$

Case II

Subcase (i) n = 1
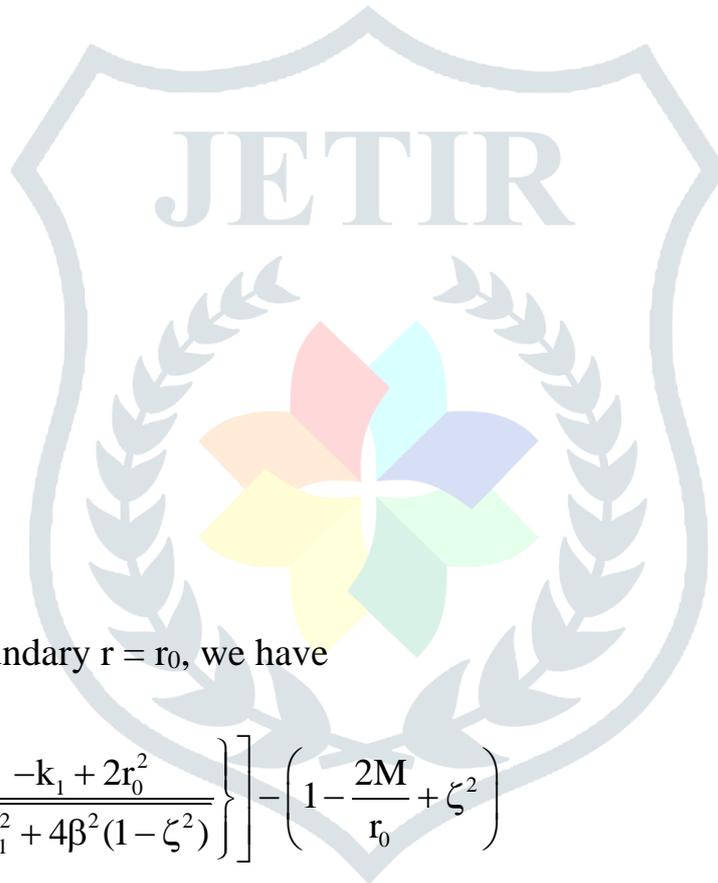
In this case at the boundary r = r_0, we have

(4.14) $\quad \exp\left[\sin^{-1}\left\{\dfrac{-k_1+2r_0^2}{\sqrt{k_1^2+4\beta^2(1-\zeta^2)}}\right\}\right]-\left(1-\dfrac{2M}{r_0}+\zeta^2\right)$

(4.15) $\quad \beta^2\left[k_1 r_0^2-r_0^4+\beta_0^2(1-\zeta^2)\right]^{-1}=\left(1-\dfrac{2M}{r_0}+\zeta^2\right)^{-1}$

(4.16) $\quad \exp\left[\sin^{-1}\left\{\dfrac{-k_1+2r^2}{k_1^2+4\beta^2(1-\zeta^2)}\right\}\right]\dfrac{r_0^3}{k_1 r_0^2-r_0^4+\beta^2(1-\zeta^2)}=M$

With help of equations (4.14), (4.15) and (4.16), we can easily obtain

(4.17) $\quad M=\dfrac{r_0^3}{\beta^2}\sqrt{k_1 r_0^2-r_0^4+\beta^2(1-\zeta^2)}$

Since M, ζ and β are positive quantities. Thus

(4.18) $r_0^2 > k_1$

The value of r_0 is obtained by putting $p = 0$ in eqn. (3.13)

Subcase (ii) n = 3

(4.19) $\exp\left[\dfrac{1}{\sqrt{(1-\zeta^2\beta^2}}\sin^{-1}\left\{\dfrac{2r_0^2(1-\zeta^2\beta^2-k_1}{k_1^2-4\beta^2(1-\zeta^2\beta^2)}\right\}\right]$

$\left(1-\dfrac{2M}{r_0}+\zeta^2 r_0^4\right)$

(4.20) $\beta\left[\beta^2+k_1 r_0^2-(1-\beta^2\zeta^2)r_0^4\right]^{-1}=\left(1-\dfrac{2M}{r_0}+\zeta^2 r_0^4\right)^{-1}$

(4.21) $\exp\left[\dfrac{1}{\sqrt{1-\zeta^2\beta^2}}\sin^{-1}\left\{\dfrac{2r_0^2(1-\zeta^2\beta^2)-k_1}{k_1^2-4\beta^2(1-\zeta^2\beta^2}\right\}\right]$

$\dfrac{r_0^3}{\sqrt{k_1 r^2-\beta^2-r_0^4(1-\zeta^2\beta^2)}}=M+2\zeta^2 r_0^5$

By use of equations (4.19) and (4.20) in (4.21) we have

(4.22) $\dfrac{r_0^3}{\beta^2}\sqrt{\beta^2+k_1 r_2^2-(1-\zeta^2\beta^2)r_0^4}=M+2\zeta^2 r_0^5$

Again for positive M, ζ and β one has

(4.23) $k_1 > \left(1-\beta^2\zeta^2\right)r_0^2$

Further putting the value of $p = 0$ in equation (2.3.17) the value of r_0 is obtained.

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