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# Stereographic projection using Complex Analysis of one variable

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***Abstract :*** Stereographic projection is a form of mapping that projects points from a sphere onto a plane. This type of projection is notable for its ability to preserve angles, making it a conformal map. It is particularly useful in various fields, including cartography, complex analysis, and crystallography. Stereographic projection angles, making it useful in applications where the local shape of features must be preserved. Circles on the sphere are projected as circles on the plane, although great circles (which go through the North Pole) are projected as straight lines. The entire plane can be used, with points near the North Pole projecting to far points on the plane, implying the North Pole itself projects to infinity. In this paper we give mathematical representation of Stereographic projection and the where it can be applied.

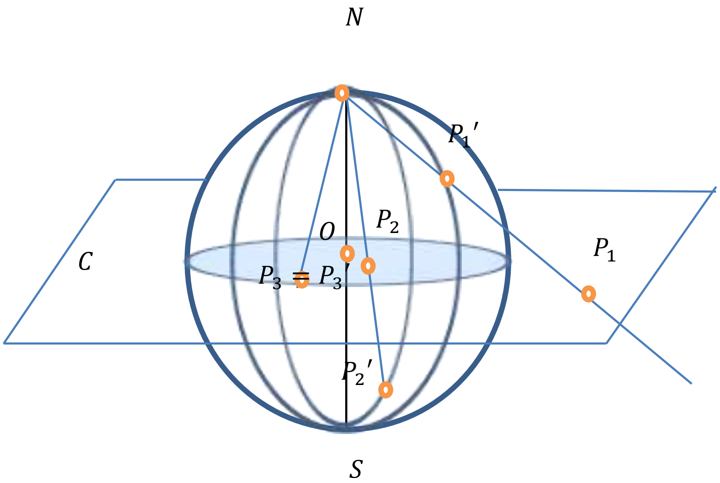
***Index Terms* – Stereographic projection; conformal map; Complex Number.**

## 1. Introduction

The concept of stereographic projection dates back to ancient Greece. The mathematician Hipparchus (circa 190-120 BC) is credited with early developments in this area, using stereographic projection in his star charts. Claudius Ptolemy (circa 100-170 AD) later adopted and extended these ideas, utilizing stereographic projection in his work "Geographia" to create maps of the known world. During the Islamic Golden Age, mathematicians like Al-Biruni (973-1048) further developed the techniques of stereographic projection, applying them in astronomy and geography. Renaissance mathematicians and cartographers such as Gerhard Mercator (1512-1594) employed stereographic projection to improve map accuracy, especially for celestial maps.

## 2. Stereographic projection

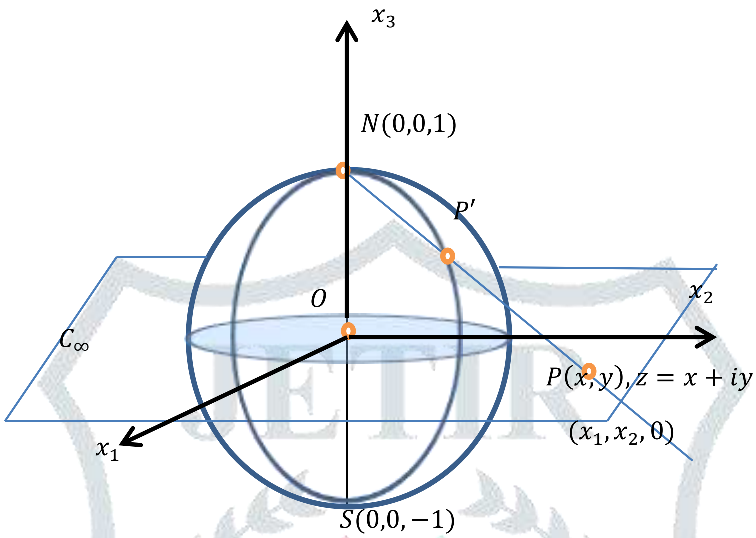
Let C be the complex plane. We draw a sphere of unit radius centred at O (z=0) and the plane cuts the sphere along a circle may be called equator. The diameter perpendicular to the complex plane meets the sphere at N and S, called North Pole and South Pole, respectively. Let $P_1$, $P_2$ and $P_3$ be three points lie on outside, inside and on the sphere in the complex plane C. The lines $NP_1$, $NP_2$ and $NP_3$ meet the sphere at points $P_1'$, $P_2'$, $P_3' = P_3$ respectively. It is evident that points $P_1'$ and $P_2'$ lie on upper hemisphere and lower hemisphere respectively. Thus a point $z$ in the complex plane lying outside or inside the circle maps onto a point lying on the upper



hemisphere or lower hemisphere respectively. A point $z$ on the circle maps onto itself. The point $z = 0$ maps onto the South Pole S. The North Pole N maps onto the "point at infinity" of the complex plane C.
The method described above for mapping of whole complex plane C onto the unit sphere is called the **stereographic projection**. The set of all points of the complex plane including the "point at infinity",∞, is called **extended complex plane**. The sphere is often called **Riemann Sphere**. In this manner on can establish a one-one and onto correspondence between the Riemann Sphere and the extended complex plane.

**3. Spherical representation of complex number or Projection of complex number on Riemann Space**

From the stereographic projection we know that every point $P(z = x + iy)$ of extended complex plane $C_\infty$ corresponds to a unique point $P'$ on the Riemann Sphere of unit radius centered at O.




We consider a three dimensional rectangular frame of reference at origin O such that
$x_1$ axis along real axis of the complex plane
$x_2$ axis along imaginary axis of the complex plane
$x_3$ axis along $ON$ perpendicular to the complex plane, where N is the North Pole of the Riemann Sphere.
Then $N$ is $(0,0,1)$ and $P$ is $(x, y, 0)$.
The equation of sphere is

$$x_1^2 + x_2^2 + x_3^2 = 1 \qquad (1)$$

The equation of the line $NP$ is

$$\frac{x_1 - 0}{x - 0} = \frac{x_2 - 0}{y - 0} = \frac{x_3 - 1}{0 - 1} = 1 - u \ (say) \qquad (2)$$

u is arbitrary parameter.
∴ a point on the straight line $NP$ is
$\big((1-u)x, (1-u)y, 1 + (1-u)(-1)\big)$ or $\big((1-u)x, (1-u)y, u\big)$.
If this point lies on the sphere (1) then
$\{(1-u)x\}^2 + \{(1-u)y\}^2 + u^2 = 1$
$or,\ (1-u)^2(x^2 + y^2) - (1 - u^2) = 0$
$or,\ (1-u)(x^2 + y^2) - (1 + u) = 0,$ considering $u \neq 1$, which refers to the point $N$
$or,\ u\{-(x^2 + y^2) - 1\} = -(x^2 + y^2) + 1$

$$or,\ u = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} = \frac{|z|^2 - 1}{|z|^2 + 1} \qquad (3)$$

$\therefore\ 1 - u = 1 - \frac{|z|^2 - 1}{|z|^2 + 1}$

$$or,\ 1 - u = \frac{2}{|z|^2 + 1} \qquad (4)$$

Hence, the spherical representation of $P$, which is $P'$, is given by

$$x_1 = \frac{2x}{|z|^2 + 1} = \frac{2Re(z)}{|z|^2 + 1},\ x_2 = \frac{2y}{|z|^2 + 1} = \frac{2Im(z)}{|z|^2 + 1} \text{ and } x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}. \qquad (5)$$

**3.1 Another form of $P'$**

We have $z = x + iy, \bar{z} = x - iy$
$\therefore z + \bar{z} = 2x, z - \bar{z} = 2iy, z\bar{z} = x^2 + y^2 = |z|^2$.
Hence from (5)

$$x_1 = \frac{z - \bar{z}}{z\bar{z} + 1},\ x_2 = \frac{z - \bar{z}}{i(z\bar{z} + 1)} \text{ and } x_3 = \frac{z\bar{z} - 1}{z\bar{z} + 1}.$$

**Note 3.1**
It should be noted that the Riemann sphere need not necessarily be formed with centre at origin and radius one. In fact a Riemann sphere can be drawn anywhere with its surface on the complex plane and of any radius.

**4. Solved examples**

**Example 4.1**

Find the projection of $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ on the Riemann sphere.

**Solution:**

Let $z = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$, then $\bar{z} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$.

$\therefore z + \bar{z} = \sqrt{2}, z - \bar{z} = -\sqrt{2}, |z|^2 = \frac{1}{2} + \frac{1}{2} = 1.$

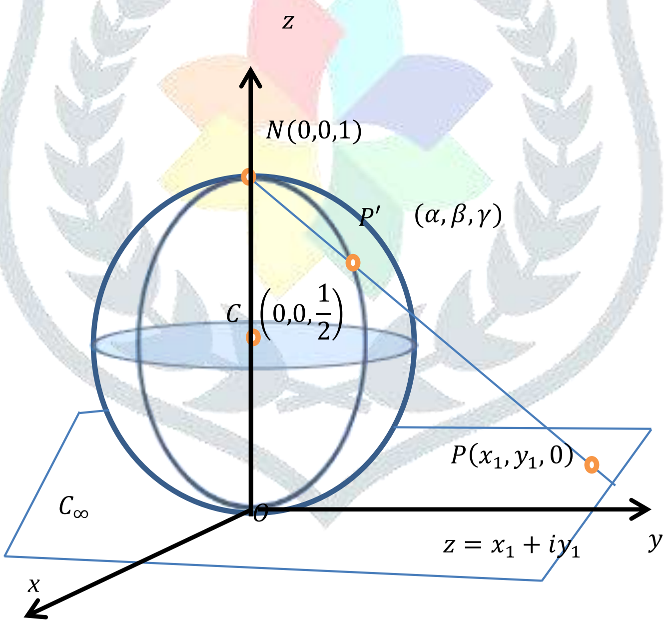
We know if $(x_1, x_2. x_3)$ be the projection of the point z of complex plane on the Riemann sphere $(x_1^2 + x_2^2 + x_3^2 = 1)$ then

$$x_1 = \frac{z + \bar{z}}{|z|^2 + 1} = \frac{\sqrt{2}}{1 + 1} = \frac{1}{\sqrt{2}}$$
$$x_2 = \frac{z - \bar{z}}{i(|z|^2 + 1)} = \frac{-\sqrt{2}\,i}{i(1 + 1)} = -\frac{1}{\sqrt{2}}$$
$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1} = \frac{1 - 1}{1 + 1} = 0$$

$\therefore$ the required stereographic projection of $z$ on the Riemann sphere is $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$. ∎

**Example 4.2**

If $z$ is a point on the complex plane and $(\alpha, \beta, \gamma)$ is the stereographic projection on the Riemann sphere $x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$ then prove that $z = \frac{\alpha + i\beta}{1 - \gamma}$ $(i = \sqrt{-1})$.




**Solution:**

The equation of the Riemann sphere is

$$x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4} \qquad (1)$$

Here the Riemann sphere is drawn with South Pole at origin O.

The centre C and North Pole N of the sphere are at $\left(0,0,\frac{1}{2}\right)$ and $(0,0,1)$ respectively.

Let the point $P$ be

$z = x_1 + iy_1 = (x_1, y_1, 0)$.

The equation of the line $NP$ is

$$\frac{x - 0}{x_1 - 0} = \frac{y - 0}{y_1 - 0} = \frac{z - 1}{0 - 1} = 1 - u\ (say), \qquad (2)$$

where u is a real parameter.

Hence any point on the line (2) is $\left((1 - u)x_1, (1 - u)y_1, u\right)$.

Let this point lies on the sphere (1). Then putting in (1) we get

$$\{(1-u)x_1\}^2 + \{(1-u)y_1\}^2 + \left(u - \frac{1}{2}\right)^2 = \frac{1}{4}$$

$$or,\ (1-u)^2(x_1^2 + y_1^2) + \left(u^2 - u + \frac{1}{4}\right) = \frac{1}{4}$$

$$or,\ (1-u)^2(x_1^2 + y_1^2) - u(1-u) = 0$$

$or,\ (1-u)(x_1^2 + y_1^2) - u = 0,$ considering $u \neq 1$, u=1 refers to the point $N$

$$or,\ u\{-(x_1^2 + y_1^2) - 1\} = -(x_1^2 + y_1^2)$$

$$u = \frac{x_1^2 + y_1^2}{x_1^2 + y_1^2 + 1} = \frac{\lambda}{\lambda + 1}, \lambda = x_1^2 + y_1^2$$

$$\therefore 1 - u = \frac{1}{\lambda + 1}$$

Since $(\alpha, \beta, \gamma)$ is the stereographic projection of $z$ on the Riemann sphere (1),

$$\therefore \alpha = (1-u)x_1 = \frac{1}{\lambda+1}x_1, \beta = (1-u)y_1 = \frac{1}{\lambda+1}y_1, \gamma = u = \frac{\lambda}{\lambda+1}$$

$$\therefore x_1 = (\lambda+1)\alpha, y_1 = (\lambda+1)\beta, 1 - \gamma = \frac{1}{\lambda+1}$$

$$\therefore x_1 = \frac{\alpha}{1-\gamma}, y_1 = \frac{\beta}{1-\gamma}$$

$\therefore z = x_1 + iy_1 = \frac{\alpha + i\beta}{1-\gamma}$. (Proved). ■

## 5. Applications of Stereographic Projection

5.1. Cartography : In map-making, stereographic projection is used to represent the Earth's surface. It is particularly useful for polar charts where distortions are minimized around the poles.
5.2. Complex Analysis : Stereographic projection maps the extended complex plane (including infinity) onto a sphere, known as the Riemann sphere. This is essential in understanding complex functions, particularly those involving Möbius transformations.
5.3. Crystallography : Crystallographers use stereographic projection to visualize the orientation of crystals. The projection helps in representing crystal lattices and planes, simplifying the analysis of crystal structures.
5.4. Geometry and Topology : In mathematics, stereographic projection helps in visualizing and proving properties of geometric shapes. For example, it aids in understanding the properties of spherical triangles and polyhedra.
5.5. Astronomy: Astronomers use stereographic projection to map the celestial sphere onto a flat surface, such as in star charts. This helps in visualizing the positions and movements of celestial bodies.

## 6. Conclusion

Stereographic projection is a powerful tool in various scientific and mathematical disciplines due to its unique properties of angle preservation and circle transformation. Whether used in map-making, complex analysis, or visualizing crystal structures, its applications are diverse and profound.

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