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A study of dissipative Systems based on LAP

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Abstract : In classical mechanics, various approaches have been formulated based on the concepts of force, energy, least action path, etc. They could be applied to figure out the motion equations for conservative systems, making them effective tools for system predictions. It is also possible to extend it to non-conservative environments. In this work, we have extended the physical system with dissipating factor prediction, with the help of the "Euler-Lagrange equation, Hamilton's principle, and the least action principle". Such an explanation argues that an object will always select the path that requires the least amount of action, irrespective of any dissipating factor. But as we move to higher dissipating factors the optimal nature of action is lost. All these make systems nonlinear with an integer dissipating factor which indirectly depends on the system and surrounding nature. A brief analysis of system dynamics was studied up to the third order of dissipation, especially for spring-mass system. It is found that dissipating variables affect not only velocity variables but also perturbed path/action integrals. The Hamiltonian is conserved in the case of non-forced oscillation. The more sensitive behavior of damping environments on a time scale is also noted.

Keywords: **Classical mechanics, Least Action Principle, Dissipative Systems, Variational Calculus**

1. Introduction

Differentiable dynamical systems are studied mathematically to understand natural time evolutions. This connection to physical reality has shaped the field. Mathematical and physical development had other different approaches to understanding reality. All these natural time evolutions produce two large classes of dynamical systems – Conservative and dissipative. Conservative systems are frictionless and dissipative systems convert "noble" energy to heat. To study conservative system, sir Isaac Newton shaped Natural Philosophy [1] and Largangian rely on variation mechanics which uses least action principle (LAP). This is very important because of variational mechanics has nontrivial conservative systems that can be observed and theoretically studied with extreme precision. Conservative and dissipative systems are equally interesting. Dissipative systems do not follow simple laws and need infinite parameters to describe them. Dissipative phenomena include motion under viscous fluids, diffusion, chemical reactions, and resistor heating etc [7]. To understand dissipative systems, different authors use even a simple dynamical system like simple pendulum, spring-mass system in a viscous medium. All of them found that such simple system becomes complex one and very difficult to predict after a certain range [5,6]. Experimental study of such system dynamics is difficult and sensitive to initial conditions. A small change in initial condition produces an orbit that deviates exponentially (at least temporarily) from the unperturbed orbit. This system's turbulence is a major theoretical physics problem as internal friction converts mechanical energy into heat, their behaviour study can help scientists in describing others' phenomena.

It can be difficult to predict how dissipative systems in nature would behave for a variety of reasons. These consist of chaotic behaviour, complicated and non-linear behaviour, sensitivity to initial conditions, incomplete information, external effects, and disturbances [2]. Despite these difficulties, scientists and researchers investigate dissipative systems and learn about their behaviour by using a variety of mathematical models, computer simulations, and actual observations [2,10,12]. Predicting the precise behaviour of dissipative systems in nature is still challenging due to the inherent complexity and uncertainty involved. The various approach for trajectory finding includes Newtonian, Lagrangian, and Hamiltonian formulation. This work focuseson the least action principle (LAP) to study dissipative system trajectory. This will be better one because it is based of energy calculation, not on individual vector quantities like force. Rayleigh and

Bateman in the year 1931 has introduces a new Lagrangian in order to account dissipation [2]. The consider Eulier-Lagrange equation is

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial D}{\partial \dot{x}} = 0 \qquad (1.1)$$

where the damping term 'D' is 12x,, a special case of Stokes's Law : fd=-mζx, where 'm' is the mass of the body and ζ is the drag constant. Due to non existence of unique Lagrangian for dissipated energy. Bateman [2] proposed a dissipative Lagrangian that was equal to the conservative Lagrangian multiplied by the factor℮ζt or

$$L = e^{\zeta t}(k - v) \qquad (1.2)$$

Even though this Lagrangian satisfies the least action principle, but it only applies to linear damping and has no physical significance [10, 13]. All this is because the principle of least action is commonly expressed by Hamilton and solely takes into consideration. In order to rephrase it for non-conservative forces, Wang and Lin in 2014 & 2015 introduced the term path optimization term with friction in the LAP approach [10,13,14]. The optimality argument fails for high damping coefficient levels, as modified action depends on damping type. In 2019 , A. Vaidya's investigated this modified least action principle for the cases of a linearly and cubically damped, nonlinear simple pendulum with dissipation [5].

This work also uses Wang and Lin in 2014 & 2015 LAP path optimization idea and A. Vaidya's 2019 modified action principle for dissipative system. The section 2 discusses modified LAP approach for dissipative systems in details. The spring-mass system is always considered as role model for understanding dissipations. In section 3 we design a spring-mass system system model based on LAP to validate wang's theory and also to improve scientific credibility of Vaidya.

A computation approach is design with damping in section 4. All this enhance the quality and firmness of the idea, ultimately leading to a more rigorous and practical understanding of the behavior of a physical system. Ultimately it is found that the behaviour of action is no longer minimum even in the case of low damping in the spring-mass system. An interesting thing to also notice in the case of Hamiltonian, which is constant with time in the damped case.

2 . LAP approach for Dissipative System

For a the nonlinear dissipating system, it is the value of damping constant which describe it into free, damping or forced oscillation. The Wang dissipative LAP approach is based on energy calculation, and rate of energy change in isolated systems with damping. The rate of energy change is introduced by friction constitutes an isolated system. One important point to be note here is that even if dissipating system is a non-conservative system, but dissipative energy on its own, energy is transferred from the dampened body to the environment, making the system as a whole energy-conservative.

In this instance, it is assumed that the Hamiltonian reflects all of the system's energy. The Hamiltonian for a dissipative system is

$$H = H_1 + H_2 + H_{in_t} \qquad (2.1)$$

Here $H_1 = k_1 + v_1$ is the Hamiltonian of the damped body, and $H_2 = k_2 + v_2 + E_d$ is the Hamiltonian of the environment and Hint is interaction energy. Ed is the energy dissipated from the body to the environment. The interaction energy, or Hint, is most important and accounts for any variations in how the environment and the body interact in close proximity to the body. These have different value and their values will decide system behavior.

In order to predict a dissipative systems, let us consider k1, v1 kinetic energy and potential energy of dynamical system in a environment with k2 , v2 is the kinetic energy and potential energy of the constituents of the environment with energy dissipation Ed . For immobile environment k2=0 ,make v2 is steady and equal to the initial potential energy of motion. For interface conditions remain unchanged, the effective Hamiltonian (2.1) becomes

$$H = k_1 + v_1 + E_d \qquad (2.2)$$

The Legendre transform, which relates the Hamiltonian and Lagrangian, is $L(q,\dot{q}) = p\dot{q} - H(q,p)$ *where* $'p'$ is the canonical momentum *and* $p\dot{x} = 2k_1$ and the Lagrangian for the system is [8, 9]

$$L = 2k_1 - H \qquad (2.3)$$

By substitution , we get

$$L = k_1 - v_1 - E_d \qquad (2.4)$$

with the corresponding action,

$$S = \int_{t_1}^{t_2} L(x,\dot{x},t)\,dt = \int_{t_1}^{t_2} (k_1 - v_1 - E_d)\,dt \qquad (2.5)$$

E_d is the energy dissipated from the body to the environment from the initial moment of motion to the time 't' due to the friction force. Hence [10, 14]

$$E_d = -\int_{x(t_a)}^{x(t_b)} f_d\big(x(\tau),\dot{x}(\tau)\big)\,d\big(x(\tau)\big) \qquad (2.6)$$

where f_d is a suitable function of time, position, and velocity. Since E_d depends not only on the current time but also on the past trajectory, E_d is a nonlocal variable and causes both the Hamiltonian and the Lagrangian as defined above to be nonlocal as well. This non-locality issue has been resolved by wang. Substituting this effective Lagrangian into the Euler-Lagrange equation gives [11]

$$\frac{d}{dt}\left(\frac{\partial k_1}{\partial \dot{x}}\right) + \frac{\partial v_1}{\partial x} + \frac{\partial Ed}{\partial x} = 0 \qquad (2.7)$$

The term $\frac{\partial E_d}{\partial x}$is the same as

$$\frac{\partial E_d}{\partial x} = -\frac{\partial}{\partial x}\int_{x_1}^{x_2} f_d\big(x(\tau),\dot{x}(\tau)\big)\,dx\,(\tau) = -f_d(t) \qquad (2.8)$$

So, equation (2.6) becomes

$$m\ddot{x} + \frac{\partial v_1}{\partial x} - f_d = 0 \qquad (2.9)$$

This shows that dependence of dissipative force on velocity can have a substantial effect on a particle's dynamics [10, 14]. The drag force is a resistive force that operates in the opposite direction of the particle's motion and is inversely proportional to some power the particle's velocity. The unknown value of velocity power make solution is tedius, as it depend on various paramaters like damping nature, time, velocity direction etc. The space and time dependence of variables make situation more complex. Different models based on various power such as the linear, quadratic or even cubic complex power models has been used to characterize the drag force.

3 . Dissipative System Model : Spring-Mass System

To investigate the optimum nature of action in dissipative and forced contexts, we have used the spring-mass system as a model. All physical system, subject to force which acts as a harmonic oscillator for tiny vibrations or disturbances. Such harmonic oscillation are common in nature, are used in many artificial devices, such as clocks and radio circuits. Such stable balance conditions under low-potential-energy solution make them vital for the dynamical system study.

In spring-mass system, mass m is attached to a spring with spring constant 'k', the other end of fixed. The potential energy Ux at x=x0 is

$$U(x) = U(x_0) + (x - x_0)\frac{dU}{dx}\Big|_{x_0} + \frac{(x-x_0)^2}{2!}\frac{d^2U}{dx^2}\Big|_{x_0} + \ldots\ldots\ldots \qquad (3.1)$$

The configuration where $U(x)$ has a minimum will often be where the system settles, but by definition, that's also where the first derivative $\frac{du}{dx}$ disappears. Additionally, physics is typically unaffected by a constant offset to potential energy. That leaves us with

$$U(x) = \frac{(x-x_0)^2}{2!}\frac{d^2U}{ax^2}\Big|_{x_0}^{+0}(x-x_0)^3 \approx \frac{1}{2}k(x-x_0)^2 \qquad (3.2)$$

We also know that most of the physical systems are dissipative in nature. It is similar to spring-mass system in damping medium. Prior Hooks law and Newton's law application to spring mass system, gives differential equation of motion as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad (3.3)$$

On using LAP approach for spring mass dissipated system and solve the equations [5, 10, 14] gives :

$$\frac{d^2x}{dt^2} + \lambda\left(\frac{dx}{at}\right)^n + w_0^2 x = \alpha\, cos\, \omega_1 t \qquad (3.4)$$

Here, λ is the damping coefficient, α is the forcing coefficient, ω_1 is the angular frequency, and n is the degree of damping. We attempt to comprehend how competing damping terms affect the system in the sections that follow. We examine Eqs. (3.4) and numerically in the sections that follow to learn more about the effects of the parameters n and. The optimal solution or ideal path will be used to describe the answer to Equation (3.4). Equation (3.4) is the harmonic equation with the addition of damping and forcing. This equation can be written as a system of differential equations assuming $x = x_1,\ \dot{x} = x_2$ So the reduced equation is

$$\dot{x}_2 = -\omega_0^2 x_1 - \lambda(x_2)^n + \alpha\, cos\, \omega_1 t \qquad (3.5)$$

These equations are solved numerically for various values λ, α, n of using MATLAB's built-in differential equation solver, ode45 and also with some help of Python 'RK45' function [11]. Further, we will show the results between various parameters and different values of the coefficient.

4. Result & Analysis of Spring mass Systems

The use of least action principle for a nonlinear spring-mass system results in a modified version of the Euler-Lagrange equation given by equation (1.1). On using eqs. (1.2), (2.4) and (2.7) and analysis, the action described for optimal path xt by the nonlinear damped spring-mass is written as follows:

$$S_0^\lambda = \int_{t_1}^{t_f}\left(\frac{1}{2}\dot{\tilde{x}}^2 - \frac{1}{2}\tilde{x}^2 + \lambda\int_{x_0(t)}^{x(t)}(\dot{\phi})^n\, d\phi\right) dS \qquad (4.1)$$

The action corresponding to deviations from the nonlinear damped spring-mass's optimal path is given as follows:

$$S_\delta^\lambda = \int_{t_1}^{t_f}\left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \lambda\int_{\lambda_0(t)}^{x(t)}(\dot{\phi})^n\, d\phi\right) dS \qquad (4.2)$$

where $x(t) = \tilde{x}(t) + \delta x(t)$ corresponds to a small perturbation to $x(t)$. With these definitions in mind, we can once again articulate the Principle of Least Action in its traditional form [5]. For $\tilde{x}(t)$ the Euler-Lagrange equation is satisfied, λ with having a non-zero value, the corresponding action, given by S_0^λ is extreme in comparison to the responses to any more deviations from the path shown S_δ^λ.

$$S_0^\lambda < S_\delta^\lambda \qquad (4.3)$$

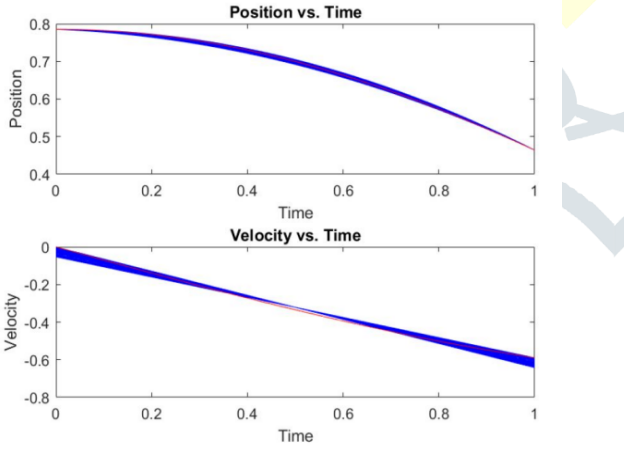
given that $\lambda < \lambda c$ where λc is a critical damping threshold, is present. After doing the mathematical calculations we arrived at the result that the least action principle is invalidated by some optimum routes because they do not match an extreme value of the action [5].

6 . Computational Result Analysis of Spring- Mass system

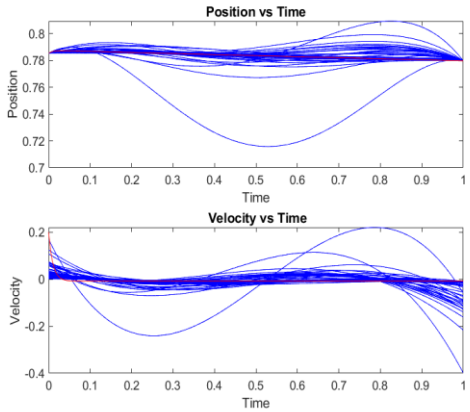
For the solution of second-order non-linear differential equations, MATLAB numerical methods like RK45 are frequently used. To check the optimality or minimum nature of action, differential equations (3.4) are solved for the various perturbed paths corresponding to the optimal path. The method of polynomial approximation [5] helps in generating various perturbed paths with fixed endpoints. Once have a complete set of data related to the path, action can generated for for all paths by solving the integral given by eq. (4.1). Further, on comparing the action for different perturbated paths including optimal paths, extremum nature can be studied for for different damping coefficients and other parameters. The Hamiltoian variation helps in study the nature of energy dissipations.

Due to the rising nonlinearity of the issue in the more general situation of the nonlinear spring-mass, numerical computation of the action is required. The optimal path is defined as the path satisfying the Euler–Lagrange equation,which in this case corresponds to eq (4.1) and eq.(4.2). The other "arbitrary paths" are created at random by adding the required amount of noise to the optimum paths. The polynomial approximations of the optimal path while preserving the initial and final positions, select one or more random path points. To illustrate one set of these randomly created pathways, we pick 100 such extra, arbitrary paths. In order to random dissipating nature of system, we notably concentrate on the instances of linear damping (n = 1) and cubic damping (n = 3) which have more physically significant [5].

For a linearly damped spring-mass system having an initial position ‘x=1’ from equilibrium with an initial velocity of zero. The varition of dissipative system pathways with damping coefficient 0.1, for unit time with step size 10^{-3} is given in Fig 1. The left side of fig 1 show variation of position and velocity with time for linear dissipative system and right side for cubic dissipative system i.e. n =3.



100 perturbed pathways in blue and an ideal path in red for λ=0.1, t=1, n=1

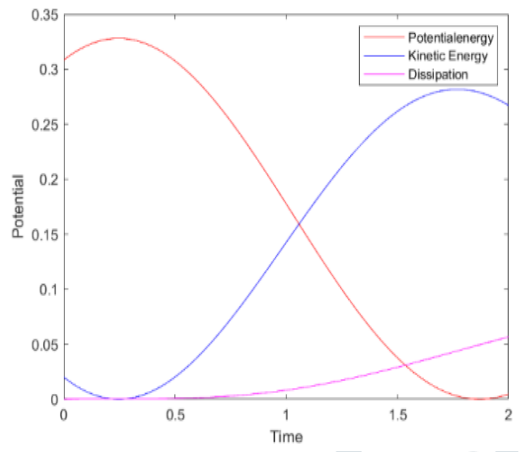


100 perturbed pathways in blue and an ideal path in red for λ=0.1, t=1, n=3

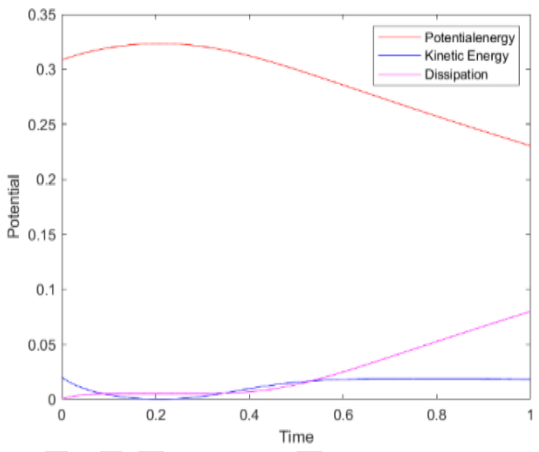
Fig 1 : Pathways VS Time for perturbed Pathways

From Fig.1, it is clear that with increasing dissipative power the variation in the position and velocity with the perturbed paths increase significantly. This suggest the strong impact on position and the nature of physical system. This can be explained by kinetic energy and potential energy. The present of dissipative also affect energy, so energy dissipation with time is also consider for investigations. The Fig. 2 depicts the behavior of potentials energy, kinetic energy and dissipation with

time for different dissipative powers. It is easily observe that dissipative energy keeps increasing with time in respect to kinetic and potential energies irrespective of any value of n. But for higher n changes are abrupt and goes on increasing with time. So due to dissipation factor in physics system, energy is getting out of the system(dissipating) rapidly with time, as expected.



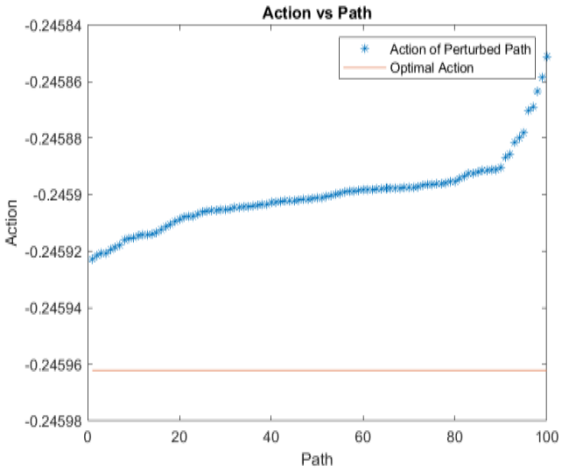
Variation of Potential, Kinetic, and dissipative energies with time for λ=0.1, t=1, n=1



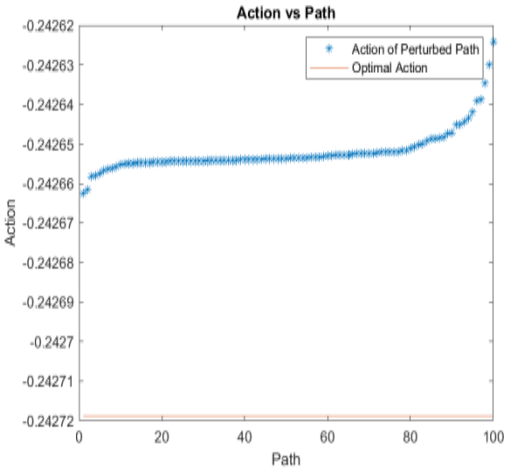
Variation of Potential, Kinetic, and dissipative energies with time for λ=0.1, t=1, n=3

Fig 2 : Variation of Potential, Kinetic, and Dissipative energies with time

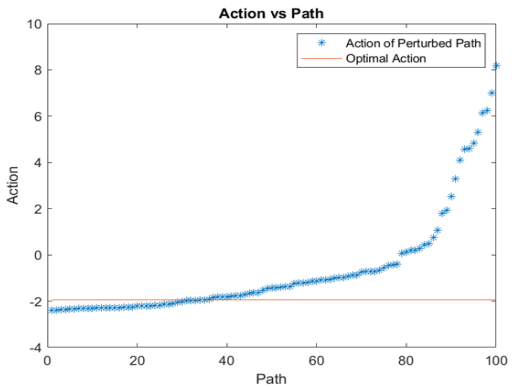
To determine the complete behavior of a dissipative system, we need to minimise the value of the integral that defines the action. Fig. 3 simply shows the behavior of action' with 'paths' for different λ, t, n values.



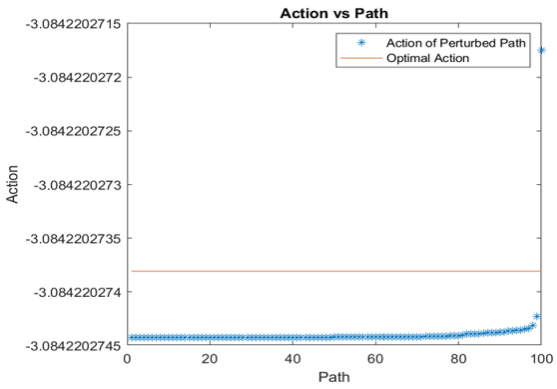
(a) Action with time for λ=0.1, t=1, n=1



(b) Action with time for =0.1, t=1, n=3



(c) Action with time for λ=10, t=1, n=3



(d) Action with time for λ=10 , t=10, n=1

Fig. 3 : Variation of Action with paths

.The variation of action with time for different λ and n is studied. For low value of dissipating coefficient Fig 3(a) and Fig 3(b) action has optimal nature, but as we move to the higher values of ' n' or "t" value can affect the nature significantly. The optimal nature completely disappears for high values of n and t, as in the case of Fig. 3(d). It is obvious that other perturbed paths have lower action value than the ideal path. Furthermore, less action than the ideal path occurs even when high λ and n have some perturbed paths. From the above analysis it can be easily observed that the nature of action is more stimulus to time than other parameters. The duration of time for which system is in damping environment has significant impact on its optimal nature.

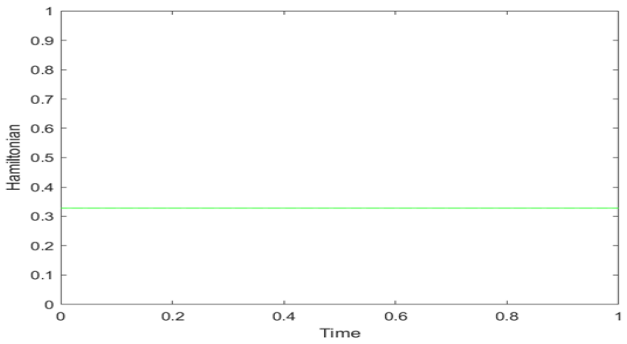


Fig 4 : Variation of Hamiltonian with time

The way Hamiltonians behave over time also does not change. Hamiltonian energy essentially refers to all of the energy present in a system, including the energy lost to the environment. This is also shown by the nature of Hamiltonain in Fig (4), as in the case of linear and cubically dissipating systems. Damping in the environment has demonstrable influence on the behaviour of physical systems, which is all it really proves. It is discovered that the extremum nature is retained, but disappears with larger damping coefficients.

6. Conclusion

Classical mechanics approaches are based on force, energy, least action path, etc. They can calculate conservative system motion equations, making them useful for system predictions. It works in non-conservative settings too. We used the "Euler-Lagrange equation, Hamilton's principle, and the least action principle" to anticipate dissipation factors in the physical system. An item will always choose the route with the least amount of action, regardless of dissipation. These make systems nonlinear with a non-integer dissipation factor that indirectly varies with the system and environment. Up to the third order of dissipation, spring-mass system dynamics were briefly investigated. Dissipating variables impact velocity and disturbed path/action integrals. Dissipative power clearly increases disturbed route position and velocity variation. This implies considerable effects on position and the physical system. Dissipative energy increases with time relative to kinetic and potential energies, regardless of n. A higher n changes abruptly and increases with time. As predicted, the physics system dissipates energy quickly. High dissipative power and time values

eliminate the optimum nature. The foregoing research shows that action is more time-sensitive than other characteristics. System performance depends on damping time. Dissipative systems preserve the Hamiltonian, although damping environments are time-sensitive.

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