



Population dynamics modeling

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Abstract : The modeling of population dynamics is of interest to several fields and has a lot of application in different fields such as demography, medicine, chemistry, economics and even astrophysics. Mathematically, there are three classical models. The Malthus model considers only the studied population. Verhulst's model takes into account the maximum population rate that an environment can contain. Finally, the Lotka-Volterra model conceives as well the dynamics of the population as of its predator. Thus, it makes it possible to study the interaction between two populations in coexistence with a particular relationship: prey-predator. In this article, we present both the differential expressions and the iterative expressions of these three models.

IndexTerms - population dynamics, Modeling, EDO, Malthus model, Verhulst model, Lotka-Volterra model.

I. INTRODUCTION

In the modeling of a dynamic system, the variable is the time t with a unit of measurement the second (s). The result of the modeling function is a number measured in thousands of inhabitants for example. However, in practice, it is considered as an average of an evolution in the size of a population per unit of time. Therefore, the result is no longer in integers but real ones. Mathematically, the temporal increase of a population $N(t)$ in a time interval k seconds is none other than the rate of variation $\frac{N(t+k) - N(t)}{k}$. The study for small variations k requires that $N(t)$ be a derivative function. Thus, the rate of variation is equated to the time derivative : $\frac{dN(t)}{dt}$. In conclusion, the function modeling the evolution over time of the size of a population is the solution of a differential equation, which is necessarily a continuous and even differentiable function.

In this work, we note

$N(t)$: the number of the population

$\frac{dN(t)}{dt}$: the rate of change

N_0 : the number of population at the initial instant $t = 0$

If, on the contrary, the time interval marking an evolution in the number of population is sufficiently large: of the order of one year, for example. Then, the mathematical model is no longer based on a differential equation but on an iterative system. The recurrence formula expresses that the population size at the n th generation T_n is a function of the population sizes of previous generations.

II. MALTHUS MODEL

2.1 Differential model

The model is

$$\frac{dN(t)}{dt} = \beta N(t) - \gamma N(t) \quad (1)$$

With β the fertility rate and γ the mortality rate. The solution to this differential equation is:

$$N(t) = \beta N_0 e^{(\beta - \gamma)t} \quad (2)$$

In program 1, we approximate the solution $N(t)$ defined by equality (2) for different parameter values of the fertility rate and the mortality rate.

Program 1 Malthus model program

```
clc();
```

```
//Discretization of the integration interval [a, b]
```

```
a=0;
```

```
b=0.4;
```

```
N=20;
```

```
h=(b-a)/N;
```

```
t=[a:h:b];
```

```
//Definition of the coefficients - initialization
```

```

delta=15;beta=5; N0=1;
//Main program
y=N0*exp((beta-delta)*t);
z=N0*exp((3*beta-delta)*t);
x=N0*exp((4*beta-delta)*t);
//Result display
disp(y);
s=t';
plot2d(s,y,-1);
plot2d(s,z,-12);
plot2d(s,x,-8);
hl=legend(['fertility < mortality'; 'fertility = mortality'; 'fertility > mortality']);
title("Malthus Model : Evolution of a population over time according to birth and mortality rates.", "fontsize",3)

```

Result 1 : The shape of the curves of $N(t)$ according to the fertility and mortality rates is illustrated in Fig. 1.

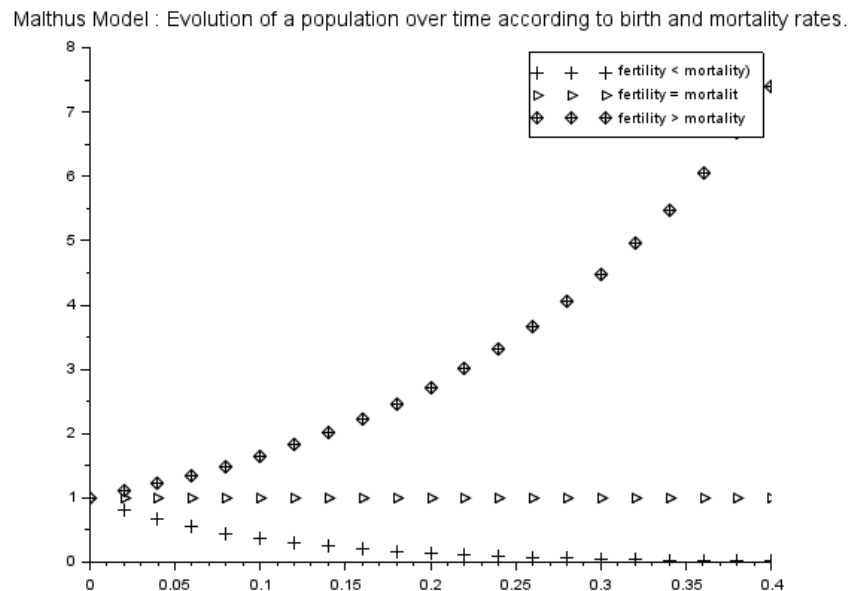


Figure 1

Interpretation 1

- If $\beta > \gamma$: the fertility rate is higher than that of mortality. We have an exponential growth in the population.
- If $\beta = \gamma$: the population number is stationary and is equal to the initial value N_0 .
- If $\beta < \gamma$: the population decreases exponentially until its extinction.

This model applies to an isolated population in a short time interval. Such is the case of a population of bacteria.

2.2 Iterative model

The iterative Malthus model has a linear form

$$T_{n+1} = rT_n$$

with $r > 0$ is the rate of variation. It is defined as the ratio between the birth rate and the death rate.

III. VERHULST'S MODEL

3.1 Logistics growth model

3.1.1 Differential model

The model is

$$\frac{dN(t)}{dt} = r N(t) \left(1 - \frac{N(t)}{k} \right) \quad (3)$$

With r the fertility rate and k the maximum number that the environment can support. The solution to this differential equation is:

$$N(t) = \frac{N_0 k e^{rt}}{k + N_0 (e^{rt} - 1)} \quad (4)$$

In program 1, we approximate the solution $N(t)$ defined by equality (4) for different parameter values of the fertility rate and the mortality rate.

Algorithm 2 Verhulst model program

```
clc();
//Discretization of the integration interval [a, b]
a=0;
b=1.4;
N=20;
h=(b-a)/N;
t=[a:h:b];
//Definition of the coefficients - initialization
delta=15;beta=5; N0=1;k=1.5;
r=beta-delta;
r1=3*beta-delta;
r2=4*beta-delta;
//main program
y=N0*k*exp(r*t)/(k-N0+N0*exp(r*t));
z=N0*k*exp(r1*t)/(k-N0+N0*exp(r1*t));
x=N0*k*exp(r2*t)/(k-N0+N0*exp(r2*t));
//Result display
disp(y);
s=t';
plot2d(s,y,-1);
plot2d(s,z,-12);
plot2d(s,x,-8);
hl=legend(['fertility < mortality';'fertility = mortality';'fertility > mortality'],'in_lower_right');
title("Verhulst Model : Evolution of a population over time according to birth and mortality rates.", "fontsize",3)
```

Result 2 The shape of the curves of $N(t)$ according to the fertility and mortality rates is illustrated in Fig. 2.

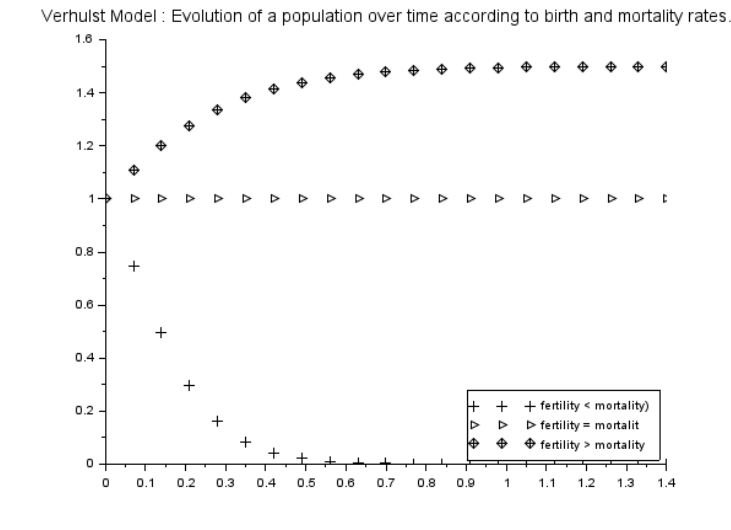


Figure 2

Interpretation 2

- If $\beta > \gamma$: the fertility rate is higher than that of mortality. this model predicts that the population will increase and converge towards its critical size k .
- If $\beta = \gamma$: the population number is stationary and is equal to the initial value N_0 .
- If $\beta < \gamma$: the population decreases exponentially until its extinction.

3.1.2 Iterative model

The iterative Verhulst model is

$$T_{n+1} = r T_n \left(1 - \frac{T_n}{k} \right)$$

with $r > 0$ is the rate of variation and k the maximum number that the environment can support.

3.2 Improvement of the model : action of a predator on a population

The model is

$$\frac{dN(t)}{dt} = r N(t) \left(1 - \frac{N(t)}{k} \right) - \frac{B N(t)^2}{A^2 + N(t)^2} \quad (5)$$

with

— A : threshold for triggering predation.

— B : controls the maximum intensity of predation.

To asize the system, we make the following changes on the units.

Variable	Unit	Dimensioned variable
t	Second (s)	$\tau = B t / A$
$N(t)$	Thousand (m)	$y(\tau) = N(t)/A$
r	1/s	$\rho = A r / B$
k	m	$q = k/A$
A	m	
B	m/s	

The model of the equation 5 becomes

$$\frac{dy(\tau)}{d\tau} = \rho y(\tau) \left(1 - \frac{y(\tau)}{q} \right) - \frac{y(\tau)^2}{1+y(\tau)^2} \quad (7)$$

This equation contains only two dimensionless parameters p and q .

IV. LOTKA-VOLTERRA MODEL

4.1 Differential model

This model is interested in the dynamic interaction of two populations in coexistence of prey-predator type. Indeed, to study the influence of predation on a population, it is necessary to include the study of the population of predators. Let's note:

$N(t)$: the number of preys at the instant t .

$P(t)$: the number of predators at the moment t .

The model is written:

$$\begin{aligned} \frac{dN(t)}{dt} &= \alpha N(t) - \beta N(t)P(t) \\ \frac{dP(t)}{dt} &= \gamma N(t)P(t) - \delta P(t) \end{aligned} \quad (8)$$

With

- $\alpha N(t)$: the rate of population growth in the absence of predators.
- $\beta N(t)P(t)$: the rate of collection of prey by predators.
- $\gamma N(t)P(t)$: the rate of increase in the predator population when $N(t)$ preys are available.
- $\delta P(t)$: the rate of decrease in the population of predators when no prey is available.

In program 3, we approximate the solution $N(t)$ and $P(t)$ of the equality (8) using the function **ode** in Scilab.

Program 1 Lotka-Volterra model program

//Definition of the coefficients - initialization

```
t0=0;
y0=[1;2];
a=4;
b=2;
c=1;
d=3;
t=0:0.1:2;
// The two EDO of the system
function [w]=LotkaVolterra(t,y)
w(1)=a*y(1)-b*y(1)*y(2);
w(2)=d*y(1)*y(2)-c*y(2);
endfunction
```

//Resolution

```
y=ode("discrete",y0,t0,t,LotkaVolterra);
```

// Result

```
subplot(2,1,1);plot2d(t,y(1,:),-1);xtitle('Evolution of populations','Time','Population');
subplot(2,1,1);plot2d(t,y(2,:), 1);
subplot(2,1,2);plot2d(y(1,:),y(2,:), 1);xtitle('Phase portrait','Prey','Predators');
```

```
//The field lines
xmax=3;
ymax=3;
dx=0.1;
dy=0.1;
X = 0:dx:xmax;
Y = 0:dy:ymax;
champ(LotkaVolterra,0,X,Y);
```

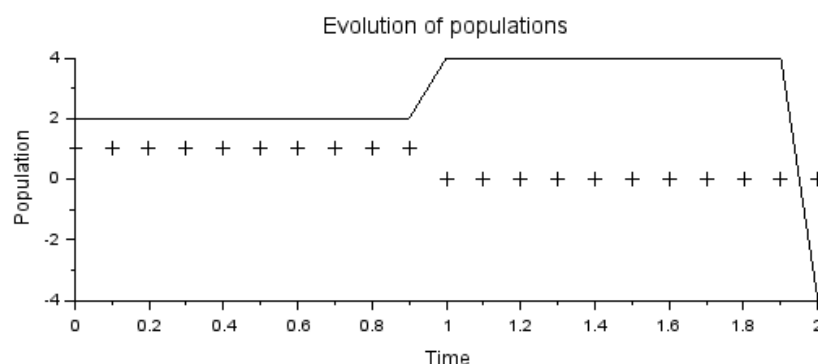


Figure 3

Interpretation 3 The evolutions of $N(t)$ and $P(t)$ predicted by the model are periodic. Indeed, this periodicity is explained by the model as follows: when the prey is abandoned, the number of predators increases. The latter have become numerous, the prey population is decreasing generating a decrease in resources. This leads to a decline in the number of predators. The latter have become rare again, the prey becomes abandoned.

4.2 Iterative model

The iterative Lotka-Volterra model is

$$\begin{aligned} T_{n+1} &= \alpha T_n - \beta T_n P_n \\ P_{n+1} &= \gamma T_n P_n - \delta P_n \end{aligned}$$

V. CONCLUSION

This article presents the different classical mathematical models of population dynamics. The differential and iterative expressions are presented for each model. In addition, numerical approximations of the solution are carried out on Scilab in each case.

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