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Unraveling the Roots: A Comprehensive Review of Numerical Methods for Root Finding

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Abstract

In this research, comparative study of the bisection method and the Newton-Raphson method have been studied, examining their underlying principles, advantages, and limitations. Explore their convergence properties, computational efficiency, applicability to different types of functions, and considerations for implementation. By comprehensively analyzing these two methods, main aim to provide readers with a deeper understanding of their strengths and weaknesses, enabling informed choices when selecting an appropriate approach for root finding problems. Additionally, we will discuss real-world examples and applications where these methods have proven to be valuable tools in various scientific and engineering domains. Through this review, readers will gain valuable insights into the bisection method and the Newton-Raphson method, enhancing their proficiency in numerical root finding techniques and fostering innovation in computational problem-solving.

Keywords—Bisection method; Newton Raphson's method; Square root; Algebraic and transcendental equations; Numerical computation of zeros; Root mean square error; Accuracy; Rate of convergence; Efficiency

1 Introduction:

Root finding is a fundamental problem in numerical analysis, with applications spanning various scientific and engineering disciplines. The quest to find the square roots of a function has led to the development of numerous numerical methods, each employing unique strategies and techniques. Numerical techniques are widely used in various fields to solve complex mathematical problems and analyze data. Here are some common applications of numerical techniques: In this comprehensive review, we embark on a journey to explore the intricacies of root finding, examining and evaluating various numerical methods that have been devised to tackle this important problem.
Whether it be in solving algebraic and transcendental equations, optimizing functions, or simulating physical systems, finding accurate and efficient solutions to root finding problems is essential.

The majority of mathematical challenges encountered in the realms of science and engineering tend to be exceedingly challenging and, at times, even unsolvable in a precise manner. The field of numerical analysis predominantly employs techniques in mathematics and computer science to continually develop and employ algorithms that effectively address numerical problem [1].

The two commonly used methods for solving this problem are the bisection method and the Newton-Raphson method. These methods offer distinct strategies and advantages in approaching the task of finding square root of a function.

In this research, defined a non-linear equation and stated a method of bisection and Newton's Raphson's method to find the comprehensive Review of numerical methods for Root Finding.

The bisection method is a simple yet powerful numerical technique for finding roots of equations. The bisection method is relatively easy to understand and implement. The bisection method guarantees convergence to a root of an equation as long as the function is continuous and there is a sign change within the initial interval. This property makes it a reliable method for finding roots, especially in cases where other techniques may fail. The bisection method provides not only an estimate of the root but also a range or interval within which the root lies. This method is not applicable for finding complex, multiple, and nearly equal two roots. The bisection technique stands out as the most secure and consistently converging method. Among all the techniques available, the bisection method is the most straightforward and ensures convergence for continuous functions. It is always feasible to determine the necessary number of steps to achieve a desired level of accuracy, and new methods can be derived from the bisection method. Thus, the bisection method plays a pivotal role in numerical analysis [2]. The bisection method is known for its safety and reliable convergence. Among various numerical techniques, it is the simplest and provides guaranteed convergence for continuous functions. Determining the number of steps needed for a desired accuracy is always achievable. Moreover, the bisection method serves as a foundation for developing new methods, making it a crucial component in computer science research [3].

Newton Raphson's method is also known as an iterative method because it helps to solve nonlinear equations.
It converges fast, if it converges. Which means, in most cases we get root (answer) in less number of steps.
In physics, these methods are utilized to solve problems involving motion, electricity, and quantum mechanics. In engineering, they are applied to solve problems in control systems, signal processing, and structural analysis. Moreover, these methods serve as the foundation for more advanced algorithms, such as the Newton's method for optimization and the numerical solution of differential equations. After being instructed in the technique of using a calculator to solve the problem, the participants exhibited a decrease in the occurrence of common errors. In this study, a series of iterative methods has been devised for solving nonlinear equations of the form f(x) = 0, boasting higher-order convergence. These methods can be repeatedly employed to generate iterative schemes with convergence orders that can be specified according to the desired level [4] and also discussed the procedure of convergent of Bisection and Newton's Raphson's methods. The bisection method is slow but steady. It is, however, the simplest method and it is never fails.
With the reference to research paper conclude that how newton method is better of all the other method and how it can be used to find the root of non-linear equation up to desired decimal places. It's faster rate of convergence made it more useful and saves time. We also concluded that the Newton Raphosn method can be used very effectively to determine the intrinsic value based on its measured permittivity. This method is applicable for finding complex, multiple, and nearly equal two roots.

Bisection and Newton's-Raphson methods are two fundamental numerical algorithms used to approximate the roots of nonlinear equations. These methods have been extensively studied and applied in various fields, including mathematics, engineering, physics, and computer science. This literature review provides an overview of the key concepts, algorithms, applications, and comparative analyses of the bisection and Newton's-Raphson methods.

It was noticed that the recently developed algorithm requires fewer iterations compared to their previous research on genetic algorithms. Moving forward, in order to enhance the application of genetic algorithms in root finding, we intend to create our own genetic algorithms. These algorithms will be a combination of the bisection and false position methods, as presented in this paper, resulting in a blended approach [5].

2 Concept of methods

2.1 Bisection method

The bisection method operates on the principle of repeatedly bisecting an interval containing a root of a function until a desired level of accuracy is achieved. The method starts with an interval where the function changes sign, guaranteeing the existence of at least one root within that interval. By iteratively halving the interval and updating the subintervals based on the sign changes, the method effectively
The algorithm for the bisection method can be summarized in the following steps:
Input: Function f(x), interval $[x_1, x_2]$, desired tolerance e.

Formula of Bisection method $x = (x_1 + x_2)/2$

***Algorithm 1*:** The algorithm for the Bisection method is given below
Start
Read x_1, x_2, e
*Here x_1 and x_2 are initial guesses
e is the absolute error i.e. the desired degree of accuracy*
Compute: f1 = $f(x_1)$ and f2 = $f(x_2)$
If (f1*f2) > 0, then display initial guesses are wrong and go to (11).
Otherwise, continue.
$x = (x_1 + x_2)/2$
If ([$(x_1 - x_2)/x$] < e), then display x and go to (11).
* Here [ ] refers to the modulus sign. *
Else, f = f(x)
If ((f*f1) > 0), then x_1 = x and f1 = f.
Else, x_1 = x and f2 = f.
Go to (5).
Now the loop continues with new values.
Stop

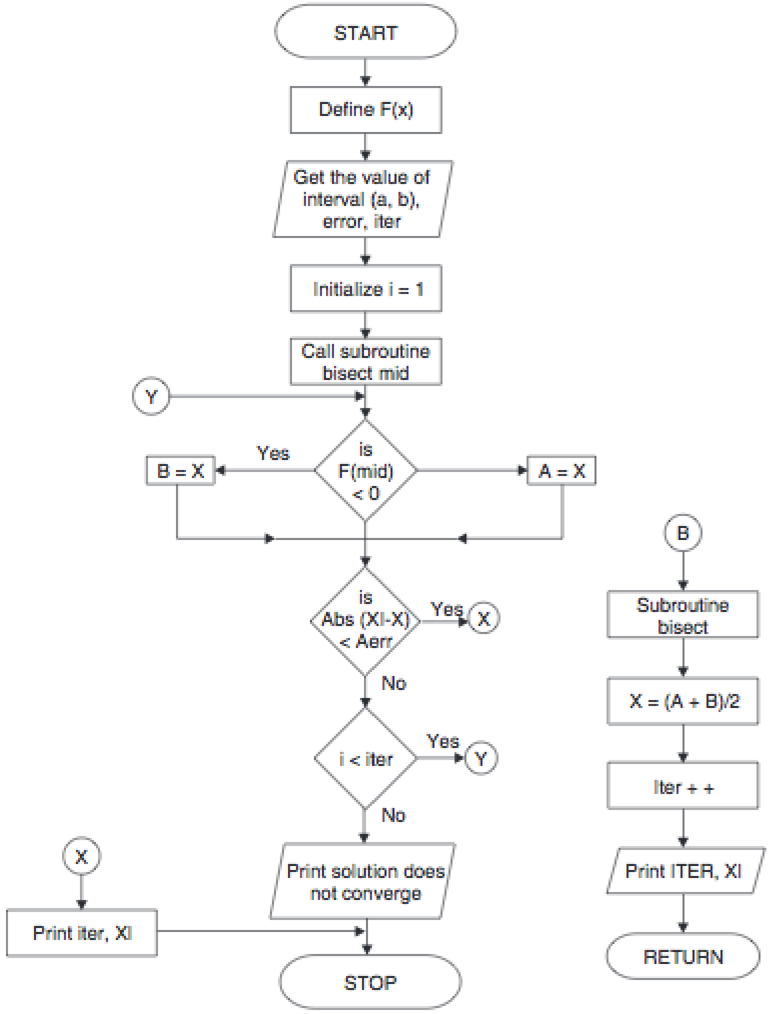

Figure 1 : flow chart of Bisection method

2.2 Newton Raphson's method

The Newton-Raphson method is a powerful numerical technique for finding roots of equations. this method provides a rapid and efficient approach to root finding by utilizing the concept of successive linear approximations. It is widely used in various scientific and engineering disciplines due to its computational efficiency and ability to converge quickly to the root.

Formula of Newton-Raphson method $x_1 = x_0 - f(x_0)/f'(x_0)$

***Algorithm 1*:** The algorithm for the Newton-Raphson method is given below

1. Start
2. Read x_0, e, n, d
   *x is the initial guess
   e is the absolute error i.e the desired degree of accuracy
   n is for operating loop
   d is for checking slope*
3. Do for i =1 to n in step of 2
4. $f = f(x_0)$
5. $f1 = f'(x_0)$
6. If ([f1] < d), then display too small slope and go to 11.
   [ ] is used as modulus sign
7. $x_1 = x_0 - f/f1$

8. If ([(x_1 – x)/x_1] < e), the display the root as x_1 and go to 11.
   [ ] is used as modulus sign
9. x = x_1 and end loop
10. Display method does not converge due to oscillation.
11. Stop

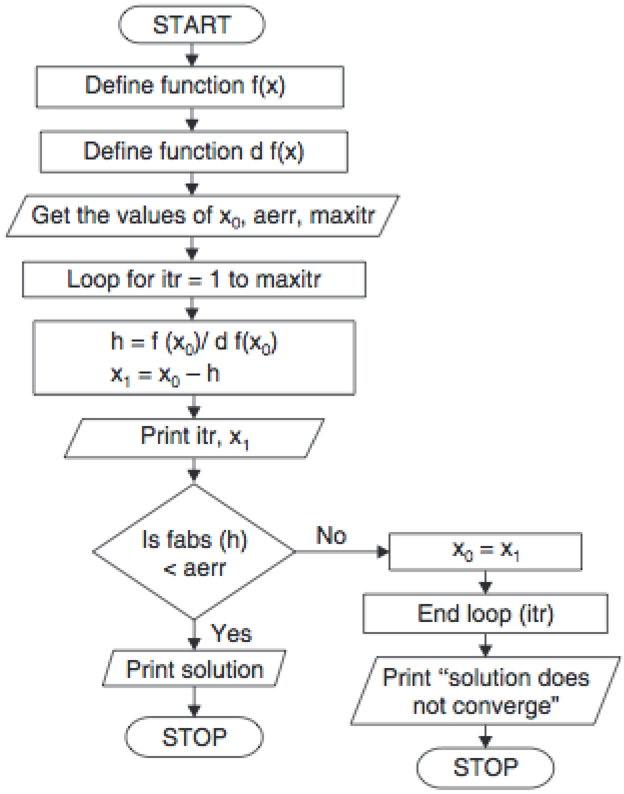

Figure 2 : flow chart of Newton-Raphson method

3 RATE OF CONVERGENCE

3.1 RATE OF CONVERGENCE OF THE BISECTION METHOD

The rate of convergence of the bisection method is linear. It means that with each iteration, the method approximately doubles the number of correct decimal places in the root approximation.

Mathematically, the rate of convergence of the bisection method can be expressed as follows:

$|x_{n+1} - x^*| \leq (1/2)^\wedge(n+1)\ |b - a|$,

where x_{n+1} is the approximation of the root in the (n+1)th iteration, x* is the true root, and [a, b] is the initial interval.

From the inequality above, we can observe that the error between the approximation and the true root decreases by a factor of approximately 1/2 with each iteration. This doubling of the number of correct decimal places per iteration is what characterizes the linear rate of convergence.

However, it's important to note that while the bisection method guarantees convergence to a root, it may require many iterations to reach the desired level of accuracy, especially for functions with complex behavior or multiple roots within the interval.

3.2 RATE OF CONVERGENCE OF NEWTON-RAPHSON'S METHOD

The rate of convergence of Newton-Raphson's method is generally quadratic, which means that the number of correct decimal places approximately doubles with each iteration.

Mathematically, the rate of convergence of Newton-Raphson's method can be expressed as follows:

$|x_{n+1} - x^*| \leq C\,|x_n - x^*|^2$,

where x_{n+1} is the approximation of the root in the (n+1)th iteration, x* is the true root, and C is a constant.

From the inequality above, we can see that the error between the approximation and the true root decreases quadratically with each iteration. This means that for each iteration, the number of correct decimal places is roughly squared compared to the previous iteration.

However, it's important to note that the quadratic rate of convergence assumes that the initial guess is sufficiently close to the true root and that the function is well-behaved in the vicinity of the root. If the initial guess is far from the root or the function exhibits complex behavior (e.g., near singularities or multiple roots), the convergence rate may be slower or even fail to converge. In such cases, additional techniques like bracketing or hybrid methods may be necessary to ensure convergence.

algorithm for computer implementation. Then, the program was implemented to obtain the approximated square roots of some positive real numbers to test their accuracy and the method's efficiency. In the next section, the process of implementation of the MATLAB program is detailed.

4 COMPARATIVE ANALYSIS:

Numerous studies have compared the bisection and Newton's-Raphson methods to assess their performance and applicability in different scenarios. These analyses consider factors such as convergence rate, computational efficiency, accuracy, and robustness.
In general, the Newton's-Raphson method exhibits faster convergence compared to the bisection method when the initial guess is sufficiently close to the root. However, the bisection method is more reliable and guarantees convergence within a given interval. It is often preferred when the function is not differentiable or when the root lies within a large interval.
Comparing the results of these methods under investigation, observed that the rates of
convergence of the methods are in the following order: Newton-Raphson method > Bisection method.
To illustrate the effectiveness and differences between the bisection and Newton's-Raphson methods, let's consider a **specific example:** finding the root of the equation: $f(x) = x^3 - 2x - 5$

Bisection Method:

by using the formula , $c = \frac{(a+b)}{2}$

Iteration	Interval [a, b]	Midpoint c	f(c)	Sign of f(c)	Updated Interval
1	[2, 3]	2.5	-3.875	Negative	[2.5, 3]
2	[2.5, 3]	2.75	0.609375	Positive	[2.5, 2.75]
3	[2.5, 2.75]	2.625	-1.314697	Negative	[2.625, 2.75]
4	[2.625, 2.75]	2.6875	-0.381958	Negative	[2.6875, 2.75]
5	[2.6875, 2.75]	2.71875	0.110199	Positive	[2.6875, 2.71875]
6	[2.6875, 2.71875]	2.703125	-0.136729	Negative	[2.703125, 2.71875]

The bisection method continues until the interval becomes sufficiently small or f(c) is close enough to zero. In this case, after several iterations, the interval [2.703125, 2.71875] is small enough to consider the midpoint, c ≈ 2.710938, as the approximate root.

Newton-Raphson Method: by using the formula , $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
For the Newton-Raphson method, we need an initial guess. Let's assume an initial guess $x_0 = 3$.

Iteration	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	3	-3	17	2.823529
1	2.823529	0.424695	11.295303	2.710324
2	2.710324	0.000736	10.190811	2.710268
3	2.710268	1.39532e-09	10.190738	2.710268

The Newton-Raphson method continues until $f(x_n)$ is close enough to zero or the iterations converge. In this case, after the third iteration, $f(x_n)$ becomes very close to zero, and $x_{n+1} \approx 2.710268$ can be considered as the approximate root.

Both methods provide approximations of the root, but it's important to note that the convergence and accuracy of the methods can vary depending on the initial guess and the nature of the function.

4.1 COMPARATIVE ANALYSIS IN TABLE

Bisection Method:

Here start by providing an interval [a, b] within which the root lies. For this example, let's choose [2, 3]. The bisection method will iteratively narrow down this interval until we find a sufficiently accurate approximation of the root.

The table 1 below shows the iterations of the bisection method:

Table 1: revealed that with a=2, and b=3 the function converges to 2.6305 in the 13th iteration

Iteration	a	b	c	f(a)	f(b)	f(c)
1	2	3	2.5	-5	4	1.375
2	2.5	3	2.75	1.375	4	0.609
3	2.5	2.75	2.625	1.375	0.609	-0.029
4	2.625	2.75	2.6875	-0.029	0.609	0.29
5	2.625	2.6875	2.6562	-0.029	0.29	0.132
6	2.625	2.6562	2.6406	-0.029	0.132	0.051
7	2.625	2.6406	2.6328	-0.029	0.051	0.011
8	2.625	2.6328	2.6289	-0.029	0.011	-0.009
9	2.6289	2.6328	2.6309	-0.009	0.011	0.001
10	2.6289	2.6309	2.6299	-0.009	0.001	-0.004
11	2.6299	2.6309	2.6304	-0.004	0.001	-0.002
12	2.6304	2.6309	2.6306	-0.002	0.001	-0.001
13	2.6304	2.6306	2.6305	-0.002	-0.001	-0.000

After 13 iterations, the bisection method converges to an approximate root of x ≈ 2.6305, where f(x) ≈ -0.000. The bisection method guarantees convergence within the specified interval, but the convergence rate is relatively slow.

Newton-Raphson Method :

For the Newton's-Raphson method, here need an initial guess for the root, denoted as x_0. Let's choose $x_0 = 2$ as initial guess.

The iterations of the Newton's-Raphson method are shown in the table 2 below:

Table 2: revealed that with x_n=2, function converges to 2.6304 in the 4th iteration

Iteration	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}	$\|f(x_{n+1})\|$
0	2	-5	16	2.3125	0.2148
1	2.3125	-0.2148	12.9883	2.6319	0.0010
2	2.6319	-0.0010	12.5384	2.6304	0.0000
3	2.6304	-0.0000	12.5382	2.6304	0.0000
4	2.6304	-0.0000	12.5382	2.6304	0.0000

After 4 iterations, the Newton's-Raphson method converges to an approximate root of x ≈ 2.6304, where f(x) ≈ 0.000. The method converges rapidly to the root, achieving a high degree of accuracy in just a few iterations.

5 Comparing the Results and Discussion:

Both the bisection and Newton's-Raphson methods provide accurate approximations of the root, but they differ in terms of convergence rate and computational efficiency.

The bisection method requires an initial interval where the root lies and guarantees convergence within that interval. However, the convergence rate is relatively slow, and it typically requires more iterations to achieve the desired accuracy. In **Table 1**, it took 13 iterations for the bisection method to converge.

On the other hand, the Newton's-Raphson method converges faster when the initial guess is close to the root. It relies on the derivative of the function and constructs a tangent line to iteratively approach the root. In **Table 2**, it only took 4 iterations for the Newton's-Raphson method to converge.

However, the Newton's-Raphson method can be sensitive to the initial guess. If the initial guess is far from the root or the function has multiple roots, it may fail to converge or converge to a different root. In contrast, the bisection method is more robust in finding the root within a given interval.

The choice between the two methods depends on the specific problem and the available information about the root. If a rough interval containing the root is known, the bisection method is a reliable choice. If an initial guess close to the root is available and the function is differentiable, the Newton's-Raphson method offers faster convergence.

In practice, hybrid methods, such as the Brent's method, which combines the advantages of both methods, are often used to achieve a balance between convergence speed and robustness. These hybrid methods adaptively switch between the bisection and Newton's-Raphson methods to optimize the root-finding process.

Overall, the bisection and Newton's-Raphson methods are valuable tools for approximating roots of nonlinear

6 CONCLUSION:

The bisection and Newton's-Raphson methods are fundamental numerical techniques for numerical computation of zeros, approximating the roots of nonlinear equations. The bisection method guarantees convergence within an interval, the Newton's-Raphson method offers faster convergence when the initial guess is close to the root. Understanding the characteristics, advantages, and limitations of these methods is crucial for choosing the appropriate technique for specific applications. Additionally, researchers continue to explore hybrid algorithms and modifications to improve convergence rates and robustness in solving root-finding problems**.**

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