



Convolution Structure of Quaternion Mellin-Wavelet Transform

Vidya Sharma* Nilesh Bhongade**

*Head, Dept. Of Mathematics, Smt. Narsamma Arts, Commerce and Science College, Amravati (MS), India

**Smt. Narsamma Arts, commerce and Science College, Amravati (MS), India

Abstract:

A quaternion is a four- element vector that can be used to encode any rotation in a 3D coordinate system. The quaternion are a number system that extends the complex numbers. A quaternion with zero real part is called a pure quaternion with unit modulus ia called a unit quaternion. The imaginary part of a quaternion has three components and may be associated with a 3-space vector.

The aim of our work is the convolution structure of Mellin-Wavelet transform. Also we have proved some basic properties like linear, shifting, distributive, conjugation property for the convolution of Mellin-Wavelet transform.

Keywords: Mellin transform, Wavelet transform, Testing function space, Mellin-Wavelet transform, quaternion Mellin-Wavelet transform.

1. Introduction

The discovery of the quaternion is one of the most well documented discoveries in Mathematics. In general, it is very rare that the date and location of a major mathematical discovery are known. In the case of quaternion, we know that they were discovered by the Irish mathematician, William Rowan Hamilton on october16th, 1843.

The ideas of this calculus of quaternion, as distinguished from its operations and symbols, are fitted to be of the greatest use in all parts of science.

The Quaternion algebra over R denoted H is an associative, non-commutative

$$H = \{q = q_0 + q_1\bar{i} + q_2\bar{j} + q_3\bar{k}, q_0, q_1, q_2, q_3 \in R\}$$

and multiplication laws:

$$ij = -ji = k, jk = -kj = i, ki = -ik = j, i^2 = j^2 = k^2 = ijk = -1$$

Quaternion play an important role in animation field because it compose rotation nicely and mainly it gives spherical interpolation[1]. In this paper we discussed the definition of quaternion Mellin-Wavelet transform, the convolution structure and some properties of quaternion Mellin-Wavelet transform.

2. Conventional Mellin-Wavelet Transform

The conventional Mellin-Wavelet transform is defined as

$$\text{MW}_\Psi\{f(t, x)\} = \text{MW}_\Psi F(p, a, b) = \int_0^\infty \int_{-\infty}^\infty f(t, x) K(t, x, p, a, b) dt dx$$

where $K(t, x, p, a, b) = \frac{1}{|a|^{1/2}} t^{(p-1)} e^{i\pi(\frac{x-b}{a})^2}$

3. Testing Function Space $\text{MW}_{\Psi,p,a,b}$:

An infinitely differentiable complex valued smooth function $\Phi(t, x, p, a, b)$ define over $-\infty < x < \infty$, $0 < t < \infty$ with parameter p, a, b is said to belong to $\text{MW}_{\Psi,p,a,b}$ for each $m, n \in R^2$, if

$$\gamma_{m,n,q,k} \Phi(t, x) = \sup_I |\xi_{m,n}(t) t^{q+1} D_t^q D_x^k \Phi(t, x)| \\ < \infty$$

Where $q, k = 0, 1, 2, 3 \dots$

$$\xi_{m,n}(t) = \begin{cases} t^{-m}, & 0 < t \leq 1 \\ t^{-n}, & 1 < t < \infty \end{cases}$$

Now we have proved the kernel of Mellin- wavelet transform belongs to the space $\text{MW}_{\Psi,p,a,b}$.

4. Quaternion Mellin-Wavelet Transform

Quaternion Mellin-Wavelet transform is defined as

$$MW_\Psi^{i,j}(p, a, b, q, a, d)$$

$$= MW_\Psi^{i,j} f(t, l, x, y)(p, a, b, q, a, d)$$

$$= \int_0^\infty \int_{-\infty}^\infty h(t, x) K_i(t, x, p, a, b) K_j(l, y, q, a, b) dt dx$$

Where $K_i(t, x, p, a, b) = \frac{1}{|a|^{1/2}} t^{(p-1)} e^{i\pi(\frac{x-b}{a})^2}$

$$K_j(l, y, q, a, b) = \frac{1}{|a|^{1/2}} l^{(q-1)} e^{i\pi(\frac{y-d}{a})^2}$$

5. Convolution Structure of Quaternion Mellin-Wavelet transform

Statement: For any real, scalar or complex signal $f(t, x)$ and convolution kernel $g(t, x)$ and $h(t, x) = (f * g)(t, x)$. Also F, G and H denote Mellin-Wavelet transform of f, g & h respectively.

$$MW_{\Psi}^{i,j} f(t, l, x, y)(p, a, b, q, a, d) = \frac{1}{|a|^2} F(u, v, p, a, b) G(m, n, q, a, a, d)$$

Where ' * ' is the Quaternion Mellin-Wavelet transform operator.

Proof: $MW_{\Psi}^{i,j} f(t, l, x, y)(p, a, b, q, a, d)$

$$\begin{aligned} &= \int_0^\infty \int_{-\infty}^\infty h(t, x) K_{i,j}(t, l, x, y) (p, a, b, q, a, d) dt dx \\ &= \int_0^\infty \int_{-\infty}^\infty \frac{1}{|a|^{1/2}} t^{(p-1)} e^{i\pi(\frac{x-b}{a})^2} \frac{1}{|a|^{1/2}} l^{(q-1)} e^{i\pi(\frac{y-d}{a})^2} h(t, x) dt dx \\ &= \int_0^\infty \int_{-\infty}^\infty \left\{ \frac{1}{|a|^{1/2}} t^{-(p-1)} e^{-i\pi(\frac{x-b}{a})^2} \frac{1}{|a|^{1/2}} l^{-(q-1)} e^{-i\pi(\frac{y-d}{a})^2} \right\} \left[\int_0^\infty \int_{-\infty}^\infty f(u, v) g(t-u, x-v) du dv \right] dt dx \\ &= \frac{1}{|a|^2} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{|a|^{1/2}} f(u, v) u^{(p-1)} e^{i\pi(\frac{v-b}{a})^2} \\ &\quad \frac{1}{|a|^{1/2}} g(t-u, x-v) (t-u)^{(q-1)} e^{i\pi(\frac{(x-v)-d}{a})^2} du dv dt dx \end{aligned}$$

put $t - u = m$ and $x - v = n$

$dt = dm$ and $dx = dn$

$$\begin{aligned} &= \frac{1}{|a|^2} \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty \frac{1}{|a|^{1/2}} f(u, v) u^{(p-1)} e^{i\pi(\frac{v-b}{a})^2} \\ &\quad \frac{1}{|a|^{1/2}} g(m, n) m^{(q-1)} e^{i\pi(\frac{n-d}{a})^2} du dv dm dn \\ &= \frac{1}{|a|^2} \left\{ \left(\int_0^\infty \int_{-\infty}^\infty \frac{1}{|a|^{1/2}} f(u, v) u^{(p-1)} e^{i\pi(\frac{v-b}{a})^2} du dv \right) \right. \\ &\quad \left. \left(\int_0^\infty \int_{-\infty}^\infty \frac{1}{|a|^{1/2}} g(m, n) m^{(q-1)} e^{i\pi(\frac{n-d}{a})^2} dm dn \right) \right\} \quad (5.1) \\ &= \frac{1}{|a|^2} \left\{ \int_0^\infty \int_{-\infty}^\infty f(u, v) K_{i,j}(u, v, p, a, b) du dv \right\} \end{aligned}$$

$$\left\{ \int_0^{\infty} \int_{-\infty}^{\infty} g(m, n) K_{i,j}(m, n, q, a, d) dm dn \right\}$$

$$= \frac{1}{|a|^2} F(u, v, p, a, b) G(m, n, q, a, d)$$

$$\therefore MW_{\Psi}^{i,j} f(t, l, x, y)(p, a, b, q, a, d)$$

$$= \frac{1}{|a|^2} F(u, v, p, a, b) G(m, n, q, a, a, d) \quad (5.2)$$

6. Properties of Quaternion Mellin-Wavelet Transform:

6.1 Linearity Property:

i) $(A_1 f + A_2 f) * h = A_1(f * h) + A_2(g * h)$

ii) $h * (A_1 f + A_2 f) = A_1(h * f) + A_2(h * g) \text{ where } A_1, A_2 \in H$

Proof: Consider,

$$\text{L.H.S.} = (A_1 f + A_2 f) * h$$

$$= \beta * h \quad \text{where } \beta = A_1 f + A_2 f$$

By using the equation (5.1) and (5.2), we get

$$\text{L.H.S.} = \beta * h$$

$$= C^2 \beta(u, v, p, a, b) h(y, z, q, a, d)$$

$$= C^2 \left\{ C \int_0^{\infty} \int_{-\infty}^{\infty} u^{(p-1)} e^{iC_1(v-b)^2} \beta(u, v) du dv \right\}$$

$$\left\{ C \int_0^{\infty} \int_{-\infty}^{\infty} y^{(q-1)} e^{iC_1(z-d)^2} h(y, z) dy dz \right\}$$

$$\text{where } C = \frac{1}{|a|^{1/2}} \text{ and } C_1 = \frac{\pi}{a^2}$$

$$= C^2 \left\{ C \int_0^{\infty} \int_{-\infty}^{\infty} \beta(u, v) K_{i,j}(u, v, p, a, b) du dv \right\}$$

$$\left\{ C \int_0^{\infty} \int_{-\infty}^{\infty} h(y, z) K_{i,j}(y, z, q, a, d) dy dz \right\}$$

$$\begin{aligned}
&= C^2 \left\{ C \int_0^\infty \int_{-\infty}^\infty [A_1 f(t, s) + A_2 g(t, s)] K_{i,j}(u, v, p, a, b) du dv \right\} \\
&\quad \left\{ C \int_0^\infty \int_{-\infty}^\infty h(y, z) K_{i,j}(y, z, q, a, d) dy dz \right\} \tag{6.1.1}
\end{aligned}$$

Consider, R.H.S. = $A_1(f * h) + A_2(g * h)$

By using equation (5.1) and (5.2), we get

$$\begin{aligned}
&= C^2 [A_1(F * h) + A_2(G * h)] \\
&= C^2 \left\{ C^2 A_1 \left[\int_0^\infty \int_{-\infty}^\infty f(t, s) K_{i,j}(u, v, p, a, b) du dv \int_0^\infty \int_{-\infty}^\infty h(y, z) K_{i,j}(y, z, q, a, d) dy dz \right] \right. \\
&\quad \left. C^2 A_2 \left[\int_0^\infty \int_{-\infty}^\infty g(t, s) K_{i,j}(u, v, p, a, b) du dv \int_0^\infty \int_{-\infty}^\infty h(y, z) K_{i,j}(y, z, q, a, d) dy dz \right] \right\} \\
&= C^4 \left\{ \int_0^\infty \int_{-\infty}^\infty h(y, z) K_{i,j}(y, z, q, a, d) dy dz \left[A_1 \int_0^\infty \int_{-\infty}^\infty f(t, s) K_{i,j}(u, v, p, a, b) du dv \right. \right. \\
&\quad \left. \left. + A_2 \int_0^\infty \int_{-\infty}^\infty g(t, s) K_{i,j}(u, v, p, a, b) dt ds \right] \right\} \\
&= C^2 \left\{ C \int_0^\infty \int_{-\infty}^\infty h(y, z) K_{i,j}(y, z, q, a, d) dy dz \right. \\
&\quad \left. C \int_0^\infty \int_{-\infty}^\infty [A_1 f(t, s) + A_2 g(t, s)] K_{i,j}(u, v, p, a, b) dt ds \right\} \tag{6.1.2}
\end{aligned}$$

From (6.1.1) and (6.1.2) result is proved

6.2 Shifting Property:

Prove that i) $(\alpha f * g) = \alpha(f * g)$

ii) $(f * \alpha g) = \alpha(f * g)$

Proof: Consider,

L.H.S. = $(\alpha f * g)$

Using (5.3.1) and (5.3.2), we get

$$= C^2 [\alpha F(u, v, p, a, b) G(y, z, q, a, d)]$$

$$= C^2 \left\{ \begin{array}{l} C \int_0^\infty \int_{-\infty}^\infty (\alpha f(u, v)) K_{i,j}(u, v, p, a, b) du dv \\ C \int_0^\infty \int_{-\infty}^\infty g(y, z) K_{i,j}(y, z, q, a, d) dy dz \end{array} \right\}$$

$$= C^2 \left\{ \begin{array}{l} \alpha \left(C \int_0^\infty \int_{-\infty}^\infty (\alpha f(u, v)) K_{i,j}(u, v, p, a, b) du dv \right) \\ \left(C \int_0^\infty \int_{-\infty}^\infty g(y, z) K_{i,j}(y, z, q, a, d) dy dz \right) \end{array} \right\}$$

$$= \alpha \{ C^2 [F(u, v, p, a, b) G(y, z, q, a, d)] \}$$

$$= \alpha(f * g)$$

$\therefore L.H.S. = R.H.S.$

Similarly we can consider

$$L.H.S. = (f * \alpha g)$$

Using (5.1) and (5.2), we get

$$= C^2 \{ [F(u, v, p, a, b) \alpha G(y, z, q, a, d)] \}$$

$$= C^2 \left\{ \begin{array}{l} \left[C \int_0^\infty \int_{-\infty}^\infty (f(u, v)) K_{i,j}(u, v, p, a, b) du dv \right] \\ \left[C \int_0^\infty \int_{-\infty}^\infty (\alpha g(y, z)) K_{i,j}(y, z, q, a, d) dy dz \right] \end{array} \right\}$$

$$= \alpha \left\{ \begin{array}{l} C^2 \left[C \int_0^\infty \int_{-\infty}^\infty f(u, v) K_{i,j}(u, v, p, a, b) du dv \right] \\ \left[C \int_0^\infty \int_{-\infty}^\infty g(y, z) K_{i,j}(y, z, q, a, d) dy dz \right] \end{array} \right\}$$

$$= \alpha [C^2 F(u, v, p, a, b) G(y, z, q, a, d)]$$

$$= \alpha(f * g)$$

$\therefore L.H.S. = R.H.S.$

6.3 Distributive Property:

Prove that $f * (g + h) = (f * g) + (f * h)$

Proof: Consider,

$$\text{L.H.S.} = f * (g + h)$$

$$= (f * w) \quad \because w = g + h$$

Using (5.1) and (5.2), we get

$$\text{L. H. S.} = (f * w)$$

$$= C^2 [F(u, v, p, a, b) w(y, z, q, a, d)]$$

$$= C^2 \left\{ \begin{array}{l} C \int_0^\infty \int_{-\infty}^\infty f(u, v) K_{i,j}(u, v, p, a, b) du dv \\ C \int_0^\infty \int_{-\infty}^\infty w(y, z) K_{i,j}(y, z, q, a, d) dy dz \end{array} \right\}$$

$$= C^2 \left\{ \begin{array}{l} C \int_0^\infty \int_{-\infty}^\infty f(u, v) K_{i,j}(u, v, p, a, b) du dv \\ C \int_0^\infty \int_{-\infty}^\infty (g + h)(y, z) K_{i,j}(y, z, q, a, d) dy dz \end{array} \right\}$$

$$= C^2 \left\{ \begin{array}{l} C \int_0^\infty \int_{-\infty}^\infty f(u, v) K_{i,j}(u, v, p, a, b) du dv \\ \left[C \int_0^\infty \int_{-\infty}^\infty g(y, z) K_{i,j}(y, z, q, a, d) dy dz + C \int_0^\infty \int_{-\infty}^\infty h(y, z) K_{i,j}(y, z, q, a, d) dy dz \right] \end{array} \right\}$$

$$= C^2 \left\{ \begin{array}{l} \left[C \int_0^\infty \int_{-\infty}^\infty f(u, v) K_{i,j}(u, v, p, a, b) du dv C \int_0^\infty \int_{-\infty}^\infty g(y, z) K_{i,j}(y, z, q, a, d) dy dz \right] \\ + \left[C \int_0^\infty \int_{-\infty}^\infty f(u, v) K_{i,j}(u, v, p, a, b) du dv C \int_0^\infty \int_{-\infty}^\infty h(y, z) K_{i,j}(y, z, q, a, d) dy dz \right] \end{array} \right\}$$

$$= C^2 \{ F(u, v, p, a, b) w(y, z, q, a, d) + F(u, v, p, a, b) H(y, z, q, a, d) \}$$

$$= C^2 [F(u, v, p, a, b) G(y, z, q, a, d)] + C^2 [F(u, v, p, a, b) H(y, z, q, a, d)]$$

$$= (f * g) + (f * h)$$

$$\therefore f * (g + h) = (f * g) + (f * h)$$

6.4 Associate Property:

Prove that $(f * g) * h = f * (g * h)$

Proof: Consider,

$$L.H.S. = (f * g) * h$$

$$= \delta * h \quad \because \delta = f * g$$

Using (5.3.1) and (5.3.2), we get

$$= C^2 \delta(u, v, p, a, b) h(m, n, q, a, d)$$

$$= C^2 \left\{ \int_0^\infty \int_{-\infty}^\infty \delta(u, v) K_{i,j}(u, v, p, a, b) du dv \right. \\ \left. \int_0^\infty \int_{-\infty}^\infty h(m, n) K_{i,j}(m, n, q, a, d) dm dn \right\}$$

$$= C^2 \left\{ C \int_0^\infty \int_{-\infty}^\infty \{(f * g)(u, v)\} K_{i,j}(u, v, p, a, b) du dv \right. \\ \left. \int_0^\infty \int_{-\infty}^\infty h(m, n) K_{i,j}(m, n, q, a, d) dm dn \right\}$$

$$= C^2 \left\{ C \int_0^\infty \int_{-\infty}^\infty \{(f(u, v)g(u, v))\} K_{i,j}(u, v, p, a, b) du dv \right. \\ \left. \int_0^\infty \int_{-\infty}^\infty h(m, n) K_{i,j}(m, n, q, a, d) dm dn \right\}$$

$$= C^3 \left\{ \int_0^\infty \int_{-\infty}^\infty \left[\int_0^\infty \int_{-\infty}^\infty f(u, v) K_{i,j}(u, v, p, a, b) du dv \right] \right. \\ \left. \int_0^\infty \int_{-\infty}^\infty g(m, n) K_{i,j}(m, n, q, a, d) dm dn \right] K_{i,j}(u, v, p, a, b) du dv \\ \left. \int_0^\infty \int_{-\infty}^\infty h(m, n) K_{i,j}(m, n, q, a, d) dm dn \right\}$$

$$\begin{aligned}
&= C^3 \left\{ \left[\int_0^\infty \int_{-\infty}^\infty \left[\int_0^\infty \int_{-\infty}^\infty f(u, v) K_{i,j}(u, v, p, a, b) du dv \right] K_{i,j}(u, v, p, a, b) du dv \right. \right. \\
&\quad \left. \left. \left(\int_0^\infty \int_{-\infty}^\infty g(m, n) K_{i,j}(m, n, q, a, d) du dv \int_0^\infty \int_{-\infty}^\infty h(m, n) K_{i,j}(m, n, q, a, d) dm dn \right) \right] \right\} \\
&= C^3 \int_0^\infty \int_{-\infty}^\infty \left\{ \left[\begin{array}{l} \left[\int_0^\infty \int_{-\infty}^\infty f(u, v) K_{i,j}(u, v, p, a, b) du dv \right] \\ \left[\int_0^\infty \int_{-\infty}^\infty g(m, n) K_{i,j}(m, n, q, a, d) du dv \right] \\ \left[\int_0^\infty \int_{-\infty}^\infty h(m, n) K_{i,j}(m, n, q, a, d) dm dn \right] \end{array} \right] \right\} K_{i,j}(u, v, p, a, b) du dv \\
&= f * (g * h)
\end{aligned}$$

6. 5 Conjugation Property:

Prove that $(\overline{f * g}) = \bar{g} * \bar{f}$

Proof: Using (5.1) and (5.2), we get

$$(f * g) = C^2 F(u, v, p, a, b) G(m, n, q, a, d)$$

$$\begin{aligned}
&= C^2 \left\{ \left[\int_0^\infty \int_{-\infty}^\infty f(u, v) K_{i,j}(u, v, p, a, b) du dv \right. \right. \\
&\quad \left. \left. \left(\int_0^\infty \int_{-\infty}^\infty g(m, n) K_{i,j}(m, n, q, a, d) dm dn \right) \right] \right\}
\end{aligned}$$

$$= C^2 \left\{ \int_0^\infty \int_{-\infty}^\infty \frac{1}{|a|^{1/2}} u^{(p-1)} e^{\frac{i\pi}{a^2}(v-b)^2} f(u, v) du dv \right\}$$

$$\left\{ C \int_0^\infty \int_{-\infty}^\infty \frac{1}{|a|^{1/2}} m^{(q-1)} e^{\frac{i\pi}{a^2}(n-d)^2} g(m, n) dm dn \right\}$$

$$= C^2 \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty f(u, v) g(m, n) u^{(p-1)} m^{(q-1)} e^{\frac{i\pi}{a^2}(v-b)^2} e^{\frac{i\pi}{a^2}(n-d)^2} du dv dm dn$$

Now $\overline{f * g}$

$$\begin{aligned}
 &= C^2 \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty f(u, v) g(m, n) u^{(p-1)} m^{(q-1)} e^{\frac{-i\pi}{a^2}(v-b)^2} e^{\frac{-i\pi}{a^2}(n-d)^2} du dv dm dn \\
 &= C^3 \int_0^\infty \int_{-\infty}^\infty m^{(q-1)} e^{\frac{-i\pi}{a^2}(n-d)^2} g(m, n) dm dn \int_0^\infty \int_{-\infty}^\infty u^{(p-1)} e^{\frac{-i\pi}{a^2}(v-b)^2} f(u, v) du dv \\
 &= C^2 \bar{g} * \bar{f} \quad \text{where } C = \frac{1}{|a|^{1/2}} \\
 &= \bar{g} * \bar{f} \\
 &\therefore (\overline{f * g}) = \bar{g} * \bar{f}
 \end{aligned}$$

Conclusion:

In this paper we have developed the new convolution structure of Quaternion Mellin-Wavelet transform. Also some basic properties of convolution structure of Quaternion Mellin-Wavelet transform are proved which will be useful in 3D coordinate system.

References:

- Sharma V. D. and Deshmukh P. B., "Convolution structure of Fractional Quaternion Fourier Transform," IJESRT, 2016, 176-182.
- Hamilton, H. R., "Elements of Quaternion", Longman, London, UK, 1864.
- Gudadhe A.S. et. al., " Scientific Reviews and Chemical Communications", 2(3), 540, 2012.
- Bhosale B., Biswas A., "Wavelet analysis of optical solitons and its energy aspects", J. Mathematics in Engineering, Science and Aerospace, 3, 15-27,2012.
- Dubey Vikas R. , "Quaternion Fourier Transform for colour Images", International Journal of computer science and Information Technologies, Vol. 5(3), 4411-4416,2014.