JETIR.ORG ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue
JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)
An International Scholarly Open Access, Peer-reviewed, Refereed Journal

Some Mathematical Models in Biomechanics

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Abstract : *The present paper provides some Mathematical models in Biomechanics. It deals with mathematical models for Airways systems of lungs and exchange of O_2 and CO_2 and energy requirements of human body along with relevant discussions.*
Key Words *: Model, lungs, peristaltic flow, ureter, human body.*

1. Introduction

Various authors and researchers have paid their wide attention towards the study of peristaltic flows in tubes and channels in Biomechanics. Such type of study is very useful and interesting in the field Modelling for Biomechanics [1– 6]

Peristaltic Motion in Biomechanics

I. Peristaltic Flows : Peristaltic means the contracting and expanding movement by which food and was to products of digestion are forced through parts of the digestion system.

We de fine the Peristaltic flow as the motion generated in the fluid which is contained in a distensible tube when a progressive wave of area contraction and expansion travels along the wall of the tube. The elasticity of the tube wall does not directly enter into our calculations, but it affects the flow through the progressive wave traveling along its length. This wave determines the boundary conditions since to no-slip condition has to be used now on a moving undulating wall surface [7–10]. Peristaltic motion is involved in

(i) Expansion and contractions (or vasomotion) of small blood vessels.
(ii) Celia transport through the ducts efferents of the male reproductive organ,
(iii) Transport of spermatozoa in cervical canal,
(iv) Transport of chime in small intestines,
(v) Functioning of ureter, and
(vi) Transport of bile.

The wide occurrence of peristaltic motion should not be surprising since it results physiologically from-neuro-muscular properties of any tubular smooth muscle.

II. Peristalitc Flow in a Channel :

We consider the flow of a homogeneous Newtonian fluid through a width 2s. Travelling sinusoidal waves are superposed on the elastic walls of the channel. Taking the x-axis along the center line of the channel and the y-axis normal to it, the equations of the walls are given by

(1.1) $$Y = \eta(X,t) = \pm a[1 + \in \cos\{(2\pi/\lambda)(X - cT)\}]$$

Where $\in$ is the amplitude ratio, λ the wavelength, and c the phase velocity of the waves. Now using (1.1) of Section (2.1), the stream function $\psi(X,Y)$ for the two-dimensional motion satisfies the equation.

(1.2) $v\nabla^4\psi = \nabla^2\psi_T + \psi_Y\nabla^2\psi_x - \psi_x\nabla^2\psi_Y$

where the velocity components are given by

(1.3) $U = \psi_Y, \quad V = -\psi_x$

Assuming that the walls have only transverse displacement at all times, we get the boundary conditions as

(1.4) $U = 0, \quad V = \pm\frac{2\pi ac\varepsilon}{\lambda}\sin\left\{\frac{2\pi}{\lambda}(x - cT)\right\}$ at $Y = \pm\eta = (X,T)$

2. Mathematical Models for Air ways system of Lungs and Exchange of O_2 and CO_2. [2, 3, 11, 12, 13]

(A) Alveolar Sacs : The air we breathe passes through the trachea which divides into two main branches, the left and right bronchi, each of which again divides into two ducts, each of which once again divides into two. This process continues for 20-22 generations, as shown in fig. 2.2. Each of the terminal duct has about 300 alveolar sacs or alveoli at the end. The diameter of each alveolar sec is between 75 and 300 microns, where a micron is equal to 10^{-6} m or 10^{-3} mm.

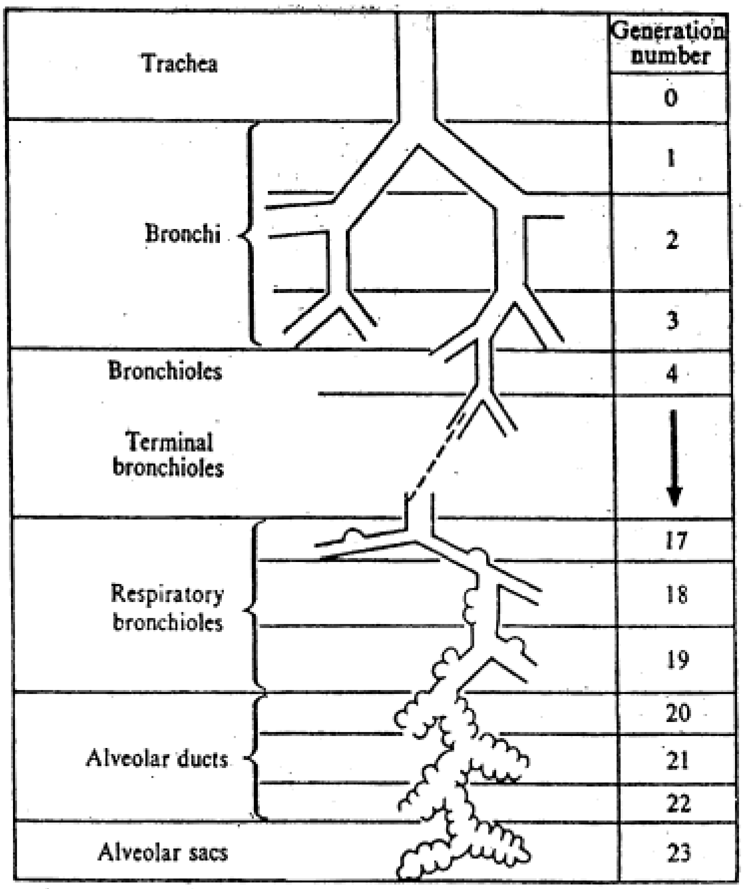

Fig. 2.2 Anatomical segments of bronchial tree

The total number of ducts is between 20^{20} and 2^{22}, but

$2^{20} = (2^{10})^2 = (1024)^2 \approx 10^6$,

Where the symbol $\approx$ is read as "is approximately equal to" or "is of the order of". Thus the total number of ducts is between 1 million and 4 millions and the total number of alveoli is between 300 millions and 1200 millions. Usually, the number is estimated to be about 300 millions.

If we take the average radius of an alveolar sac as 125 microns, its surface area is

$$4\pi\frac{25}{16}(10^{-4})^2\,m^2 \approx 2\times10^{-7}\,m^2$$

Then the total area of all alveoli is

$300 \times 10^6 \times 2 \times 10^{-7}$ m^2 = 60 m^2

It has been estimated that the total surface area is about 70 m^2. The total surfaces area of the body of an average adult human being is about 1.7 m^2 (how can you find this area?) so that, if all the alveoli are opened out and spread over the human body, they would cover the body 40 times over. If they are spread on the ground, they would cover a badminton court. The magnitude of this total surface area is important, because it is across this area that the vital exchange of O_2 in inspired air with CO_2 from venous blood takes place, and a large area is helpful in an efficient gas exchange process.

The volume of a sphere of radius r is (4/3)πr^3 , and so the volume of an alveolar sac of radius 125 microns would be about 8 × (10^{-4}) or 8 × 10^{-12} m^3 so that the total volume of all alveoli lies between 2 × 10^{-3} and 2 × 10^{-2} m^3 or between 2 × 10^3 and 2 × 10^4 cc or between 2 and 20 litres. The normal volume of alveolar sacs is about 3.5 litres. The airway segments or ducts to which these alveoli are attached have a capacity of about 1.5 litres while the purely conducting proximal air channels, namely, tracheae, bronchi, and bronchioles have a capacity of about .17 litre of air.

At every normal inspiration, an individual takes about .5 litre of 500 cc of air, of which about 350 cc go to alveoli and the remaining to 'dead space' of airway segments and conducting proximal air channels. The space is dead because it does not take part in gas exchange.

(B) Pulmonary Capillaries

The mechanism of blood supply to the lungs is as complex and interesting as that of supplying air to them. The right ventricle of the heart sends venous (deoxygenated) blood into the pulmonary artery which diyides and subdivides into billions of small tubes about 7-10 microns in diameter (see fig. 2.3). these tubes from an almost continuous network of capillaries on the alveolar walls, giving a fine mesh of small capillary segments (see Fig. 2.4). there are in all about 28 × 10^{10} capillary segments, each of about 10 microns in length and of average diameter of 8 microns. The total length of the pulmonary capillaries is equal to

$28 \times 10^{10} \times 10\mu m = 28 \times 10^{10} \times 10^{-5}$ m = 2800 km.

The total surface area of the pulmonary capillaries is

$28 \times 10^{10} \times 2\pi\,(4 \times 10^{-6}) \times 10^{-5}$ m^2 $\approx$ 70 m^2

The total volume of the pulmonary capillaries is

$28 \times 10^{10} \times 2\pi\,(4 \times 10^{-6})^2 \times 10^{-5}$ m^3 $\approx$ 140 m^2

Thus, if the pulmonary capillaries are joined end-to-end on the ground, these would cover the distance from Srinagar to Kanya Kumari. Also, the total surface area of the pulmonary capillaries is the same as the total surface of the alveoli through the volume of the pulmonary capillaries is much smaller.

The total volume of blood in the human body can be found by many methods. On method is to inject a given amount of glucose and then find the increase in the concentration of glucose in a sample of blood. For greater accuracy, we also take the metabolic absorption of glucose into account.

By all methods, the volume of blood comes out to be about 5-6 litres, which is distributed in the heart, aorta, main arteries, arterioles, capillaries, veins, and so on.

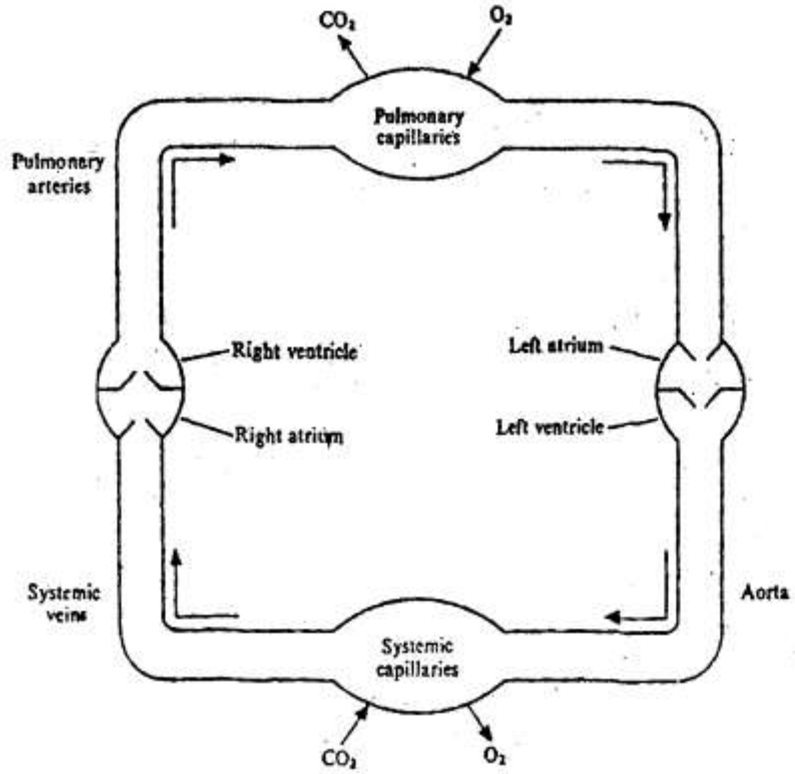

Fig. 2.3 Flow of blood and exchange of O_2 and CO_2

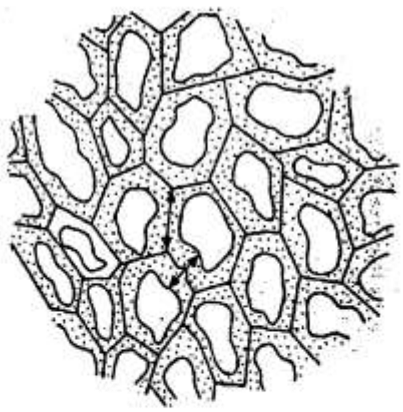

Fig. 2.4 Representation of capillary network in alveolar wall

(C) Exchange of O_2 and CO_2

Exchange of O_2 and CO_2 takes place all along the 2800 km length and 70 m^2 of surface area of the pulmonary capillaries. The exchange is goverrtned by the partial pressure of O_2 and CO_2 in the alveoli and in the pulmonary capillaries (see fig. 2.5), the diffusivity of the walls and those area of wall which are in contact with one another

O_2 159 mm Hg ↓	Atmosphere	CO_2 0.2 mm Hg ↑
↓ 100 mm Hg	Alveolar space	↑ 39 mm Hg ↑
↓ 40 mm Hg	Pulmonary capillaries	↑ 46 mm Hg

Fig. 2.5 Pressure gradients for diffusion

The inspired air is at about 760 mm of Hg pressure and it contains about 20.95 percent of oxygen so that the partial pressure of O_2 in inspired air is (20.95/100) × 760 mm of Hg ≈ 159 mm of Hg. Similarly, the partial pressure of CO_2 in inspired air is (0.04 / 100) × 760 mm of Hg = 30 mm of Hg. Table 2.1 gives partial pressures in mm of Hg for various components and the two regions.

A pressure difference of 60 mm of Hg drives O_2 from alveolar space to pulmonary capillaries. Similarly, a pressure difference of 6-7 mm of Hg drives CO_2 from pulmonary space to alveolar space. Though the pressure difference for CO_2 is much smaller than that for O_2, yet its diffusivity is equally important since solubility of CO_2 in blood plasma is about 20 times that of O_2. The flow $\dot{V}_{02}$ of oxygen from alveolar space to the pulmonary capillaries is given by

$$(2.1)\quad \dot{V}_{02} = DL_{02}(PA_{02} - PC_{02})$$

Table 2.1

Partial Pressures of Components of Inspired and Expired Air

Gas	**Inspired air**	**Expired air**	**Alveolar space**	**Venous blood**
O_2	159	116	100	40
CO_2	0.3	30	39	46
N_2	596	575	547	570
Water vapour	5.0	39	47	47
Total	760	760	733	703

Where PA_{02} and PC_{02} denote the partial pressure or tension of O_2 in alveolar space and capillaries, respectively, and DL_{02} is the diffusive capacity of the lung for oxygen.

The diffusive capacity is the reciprocal of the resistance to diffusion of an oxygen molecular from air to blood. This molecule has to travel through (i) extracellular surface lining layer (s), (ii) tissue barrier (t), (iii) blood plasma layer (p), (iv) red cell membrane and that part of the red cell protoplasm which lies between the membrane and the hemoglobin molecule to which it is going to be attached (e). Each of these surfaces offers a resistance and since these resistances are in series, we get

$$(2.2)\quad \frac{1}{DL} = \frac{1}{D_s} + \frac{1}{D_t} + \frac{1}{D_p} + \frac{1}{D_c}$$

where, D_s D_t, D_p and D_c represent the conductance in surface lining layer, tissue, plasma, and erythrocyte, respectively.

The conductance across a barrier of surface area S and thickness τ is given by

(2.3) $D = KS\,(1/\tau)$,

Where K is called the permeation coefficient and is the product of solubility and diffusion coefficients. In the present context, the thicknesses are variables and so we divide the surface area S into a large number n of small areas S/n so that the thickness of each is uniform. Then the total flow across the sheet is the sum of the flows across each element so that

(2.4) $$\dot{V}_{02} = K\frac{S}{n}\left(\frac{1}{\tau_1} + \frac{1}{\tau_2} + \ldots.\frac{1}{\tau_n}\right)(PA_{02} - PC_{02})$$

(2.5) $$\dot{V}_{02} = K\frac{S}{\tau_h}(PA_{02} - PC_{02}),$$

Where τ_h is the harmonic mean of the individual thicknesses. This discussion given the following formulae for determining D, the membrane diffusing capacity, the unit being ml O_2 min^{-1} (mm Hg^{-1})

(2.6) $$D_s(O_2) = K_s\frac{S_s + S_a}{2\tau_{hs}}, 302\times10^{-8} < K_s < 4.3\times10^{-8}$$

(Surface lining layer)

(2.7) $$D_\tau(O_2) = K_t\frac{S_a + S_c}{2\tau_{ht}}, K_t = 3.3\times10^{-8} \text{ (tissue barrier)}$$

(2.8) $$D_p(O_2) = K_o\frac{S_c + S_c}{2\pi_{hp}}, 3.2 \times 10^{-8} < K_p < 4.3 \times 10^{-8}$$

(plasma layer)

(2.9) $$\frac{1}{D_M} = \frac{1}{D_s} + \frac{1}{D_t} + \frac{1}{D_p}$$

Here, S_s, S_a, S_c and S_e are respectively the surface areas of surface lining layer, alveolar epithelium, capillary endothelium, and capillary erythrocytes, τ_{hs}, τ_{ht} and τ_{hp} are the harmonic mean thicknesses of surface lining layer, tissue barrier, and plasma.

The conductance of the erythrocytes surface is complicated by the chemical reaction of O_2 with hemoglobin and facilitated transport. A formula proposed for this is

(2.10) $D_c(O_2) = \theta V_c = \theta\,(V_c/H)$, $0.9 < \theta < 2.5$.

Here, V_c and V_e are the volumes of capillary blood and capillary erythrocytes and H is the capillary hematrocrit.

(D) Energy Requirements of Human Body [2, 14]

The cardiac output is about 5 litres per minute so that the output per year is

$5 \times 10^3 \times 60 \times 24 \times 365$ cc $= 2628 \times 10^6$ cc, the mass of which is about 2750 metric tones. Thus, in an average life time, certainly much more than 50,000 tons of blood are pumped by the heart.

The power required for driving 5000 cc of blood per minute against a pressure of 100 mm of Hg at the base of the aorta is

$$\frac{5000}{60}\times\frac{100}{10}\times 13.6\times 980 \text{ ergs/sec} \approx 10^{7} \text{ ergs/sec} = 1\text{W}$$

The work done by the heart muscles in pumping blood for one day

$10^7 \times 60 \times 60 \times 24$ ergs = 20.6 Calories = 206,000 calories.

This is only about 1 percent of the energy required by the body. In fact, the heart requires 10 times this energy for keeping it in tension. The energy required for pumping blood is a very small part of the energy required by the body.

For a man at rest, 250 ml of oxygen per minute are available for metabolic function. During exercise, the availability of this amount of oxygen increases many times over. The amount of oxygen thus obtained can be used to find the calorie requirements of the body. The reason for this is that similar amounts of heat are produced per litre of oxygen, whether or not fats, carbohydrates, or proteins are oxidized. The figures (per litre of oxygen) are: fat, 4.7 Calories, carbohydrates, 5.0 Calories, and proteins, 4.5 Calories. It is convenient to use an average value of 4.8 Calories per litre of oxygen. In one day, for a man at rest, 360 litres of oxygen are available through inspired air and this amount corresponds to 48 × 360 = 1728 Calories. For an active man, the requirement is 3000–4000 Calories per day.

3. Discussion

Comparison between Flows of Blood and Flows in Lung Airways [11, 13,14]

We now give the similarities and dissimilarities between blood flows and airways flows :

Blood is incompressible, whereas air is compressible. However, since the velocities in the lung airways are always small as compared with the speed of sound, compressibility effects can be neglected. Blood is a suspension of particles in plasma and is a nonhomogeneous non-Newtonian fluid. Air is essentially Newtonian and viscous, in addition, it contains dust particles.

Blood flow in a tube is essentially unidirectional, though some reversed flows, localized near an arterial will, may occur. However, airflow reverses its direction with inspiration and expiration and, for a very short time, the velocity can be zero when the dust particles can be deposited in the lungs.

Blood flow is essentially laminar thought there may be bursts of turbulence in the aorta. Air flow in the trachea is usually turbulent and turbulence may persist for four and five generations. In fact, if the Reynolds number is 3100 in the trachea, the intensity of turbulence is reduced to a quarter of its tracheal value in the segmental branch (Re = 1230, n = 3), and this intensity decays more rapidly afterwards.

Blood flow is pulsatile and, under normal conditions, its oscillatory boundary layer thickness is $(v/\omega)^{1/2} = 0.7$ mm. the corresponding thickness for airflow is $(15.2 / 1.57)^{1/2} = 3.1$ mm. thus the pulsatility effect is felt in blood vessels whose diameter is fives times smaller than that of the corresponding airways.

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