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Title : "ON TOPOLOGICAL GROUP AND ITS APPLICATIONS OF SEPARATION AXIOM"

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Abstract

Our aim here is to provide a characterization of separation axioms T_i (i = 0, 1, 2, 3, 4, 5) by using the concept of topology. We derive some specific properties of Lindelof spaces and include some theorems and interrelations between separability, countable basis and the Lindelof property in metric spaces. S. Willard and U. N. B. Dissanayake(2) derive topological space is quasi-Lindelof, if every closed subspace of it is weakly Lindelof. Topology plays very important role in all branches of Mathematics. An important concept introduced by J.C.Kelly(1),General Topology and Real Analysis concerns the variously modified forms of continuity and separation axioms etc. by utilizing the generalized closed sets.

Key words: (Topological space, Separation Axiom, Soft Topology Metric spaces, Lindelof spaces, open sets closed sets, Tychonoff spaces, Trennungs axiom and Paratopological groups)

1.1 Introduction

Separation axioms are a group of topological invariants that provide new methods of distinguishing between various spaces. Separation axioms shows tool for identifying how close a topological space is to being metrizable: spaces that satisfy an axiom T_i may be considered as being closer to metrizable spaces. The idea is to look how open sets in a space may be used to create "buffer zones" separating pairs of points and closed sets. Separations axioms are denoted by T_1, T_2, etc., where T comes from the German word Trennungs axiom, which just means "separation axiom". than spaces that do not satisfy T_i. A subset A of a topological space X is g-closed if the closure of A included in every open superset of A and defined a $T_{1/2}$ space to be one in which the closed sets and g-closed sets coincide. The notion has been studied extensively in recent years by many topologists. The study of g-closed sets has produced some new separation axioms. Some of these have been found useful in computer science and digital topology.

1.2 Tychonoff spaces

A topological space is called a Tychonoff space, if it is a completely regular Hausdorff space. If points may be separated from closed sets via continuous real-valued functions. In technical terms this means: for any closed set A ⊆ X and any point $x \in X/A$, there exists a real-valued continuous function $f : X \to R$ such that $f(x) = 1$ and $f|_A = 0$. Tychonoff space X and Hausdorff operation Φ, the class $\Phi(\mathcal{Z}, X)$ generated from zero sets in X by Φ has the reduction or separation property if the corresponding class $\Phi(\mathcal{F}, R)$ of sets of reals has the same property. These properties of such projective sets in X form the same pattern as the First Periodicity Theorem states for projective sets of reals: the classes Σ^1_{2n} $(\mathcal{Z}, X)$ and Π^1_{2n+1} $(\mathcal{Z}, X)$ have the reduction property while the classes $\Pi^1_{2n}(\mathcal{Z}, X)$ and $\Sigma^1_{2n+1}(\mathcal{Z}, X)$ have the separation property

1.3 Theorem

Let X be a Tychonoff space and Φ a Hausdorff operation. If $\Phi(\mathcal{F}, R)$ has the reduction (separation) property, then $\Phi(\mathcal{Z}, X)$ has the same property.

Proof

For all Tychonoff cubes $X = [0, 1]^\kappa$. For $\kappa = \omega$ this is trivial since $[0, 1]^\omega$ is Polish. For arbitrary κ, let us verify that the assumptions of $S(X) \subseteq (\mathcal{F} \cap K)(X)$ and that maps F are closed-to-one. $S(X) \subseteq (\mathcal{F} \cap K)(X)$ are met with S = Z and $Y = [0, 1]^\omega$ common for all $(A_n)_{n\in\omega}$ in S(Y), i.e., in $Z([0, 1]^\omega)$. $Z([0, 1]^\omega)$ is closed under finite intersections. If (A is in Z ([0, 1] κ), then X be a topological space and $(A_n)_{n<\omega} \in \mathcal{Z}(X)$. Then there exists a continuous map $F : X \to [0, 1]^\omega$ such that $A_n \in$ alg F for $n < \omega$. Consequently, $\Phi(A_n)_{n<\omega} \in$ alg F for any Hausdorff operation Φ gives a continuous map $F : [0, 1]^\kappa$ such that $\Phi(A_n)_{n\in\omega} \in$ alg F. As $[0, 1]^\kappa$) is compact Hausdorff, $Z([0, 1]^k)$, ⊆ $(F \cap K)([0, 1]^\kappa) = K([0, 1]^\kappa$ and moreover, F is perfect, whence it is easy to see that F and F^{-1} preserve Z. Therefore, once we have separation in $\Phi(Z, [0, 1]^\omega)$, or equivalently, in Φ(F, R), we apply $S(X) \subseteq (\mathcal{F} \cap K)(X)$, thus getting the same property in $\Phi(Z , [0, 1]^k)$. Let X be an arbitrary Tychonoff space. Pick any κ such that X may be identified with a subspace of $[0, 1]^\kappa$. Then Φ(Z , X) has reduction whenever $\Phi(Z, [0, 1]^\kappa)$. Y be Tychonoff and X ⊆ Y. Then Z (X) = Z (Y) ↾ X and consequently, for any Hausdorff operation Φ, and hence, whenever Φ(F, R) has proved.

1.4 Separation Axiom in $\pi_1^{qtop}(X, x_0)$

The fundamental groups of complicated space X is said to be π_1-shape injective if the canonical homomorphism $\psi : \pi_1(X, x_0) \rightarrow \check{\pi}_1(X, x_0)$ from the fundamental group to the first shape group is injective. If ψ is injective, then $\pi_1(X, x_0)$ may understood as a subgroup of $\check{\pi}_1(X, x_0)$, which is an inverse limit of discrete groups. If ψ is not injective, then shape theory fails to distinguish some elements of $\pi_1(X, x_0)$. There is still hope that elements of $\pi_1(X, x_0)$ which are indistinguishable by shape, might be distinguished by the quotient topology of $\pi_1^{qtop}(X, x_0)$. This possibility is one motivation for considering separation axioms in quasi-topological fundamental groups.

It is known that every T_0 topological group is Tychonoff. Since $\pi_1^{qtop}(X, x_0)$ need not be a topological group, if one can extend separation axioms within $\pi_1^{qtop}(X, x_0)$ in a similar fashion. We further relate Hausdorff properties of the topological space X and separation axioms in $\pi_1^{qtop}(X, x_0)$ by theorem.

1.5 Theorem

If X is locally path connected and H is a closed subgroup in $\pi_1^{qtop}(X, x_0)$ has the unique path lifting property.

Proof

Let C is a subset of $\pi_1(X, x_0)$ and $C \neq \pi_1(X, x_0)$. If C is closed in $\pi_1^{qtop}(X, x_0)$, then X is homotopically path-Hausdorff relative to C. If X is locally path connected and homotopically path-Hausdorff relative to C, then C is closed in $\pi_1^{qtop}(X, x_0)$.

Let H is closed in $\pi_1^{qtop}(X, x_0)$ with the condition that X is homotopy path-Hausdorff relative to H. Let us suppose $\alpha \in \mathcal{P}(X, x_0)$ is a path such that there is a lift $\beta : [0, 1] \rightarrow \tilde{X}_H, \beta(t) = [\beta_t]_H$ of α such that $\beta(0) = \tilde{\alpha}^{st}(0) = \tilde{X}_H$ but $\beta(1) = [\beta_1]_H \neq [\alpha]_H = \tilde{\alpha}^{st}(1)$. We modify that X is not homotopically path Hausdorff relative to H. $[\alpha . \beta_1^{-1}] \notin H$ and consider any partition $0 = t_0 < t_1 < \ldots. < t_n = 1$ and path connected open sets $U_1, \ldots., U_n$ such that $\alpha([t_{i-1}, t_i]) \subset U_i$. Let us consider a fixed value of i and observe that $B_H([t_1]_H, U_n)$ is an open neighborhood of $\beta(t)$ for each $t \in [t_{1-1}, t_i]$. Therefore there is a subdivision $t_{i-1} = s_0 < s_2 < \ldots s_m = t_i$ such that $\beta([s_{j-1}, s_j]) \subseteq B([\beta s_{j-1}]_H U_i)$ for each $j = 1, \ldots .m$. The specific path $\delta_j : [0, 1] \rightarrow U_i$ from $\alpha(s_{j-1})$ to $\alpha(s_j)$ such that $[\beta s_{j-1} . \delta_j]_H [\beta s_j]_H$.

The concatenation $\gamma_i = \delta_1, \delta_2 \ldots \delta_m$ is a path in U_i from $\alpha(t_{i-1})$ to $\alpha(t_i)$. Since $g_j = [\beta s_{j-1} . \delta \ \beta^- s_j] \in H$ for each j, we have

$$h_i = [\beta_{t_{i-1}} . \gamma_i . \beta^{-}_{t_1}] = g_1 g_2 \dots . g_m \in H \quad \dots (1.1)$$

Let γ be the path defined a γ_i on $[t_{i-1}, t_i]$. Then $\gamma([t_{i-1}, t_i]) \subseteq U_i$ and $\gamma(t_i) = \alpha(t_i)$. now $[\beta_0 . \gamma . \beta^{-}_1] = h_1 h_2 \dots . H_n \in H$ and $[\beta_0] \in H$. Thus $[\gamma. \beta^{-}_1] \in H$, that X is unique homotopy path-Hausdorff relative to H. Hence, theorem is proved.

1.6 Relation between Topology and soft bitopological space

Let (X, τ_1, τ_2, E) be a soft bitopological space over X and Y be a non-empty subset of X. Then $\tau_{1Y} = \{({}^Y F, E) : (F, E) \in \tau_1\}$ and $\tau_{2Y} = \{({}^Y G, E) : (G, E) \in \tau_2\}$ are said to be the relative topologies on Y . Then $\{Y, \tau_{1Y}, \tau_{2Y}, E\}$ is called a relative soft bitopological space of (X, τ_1, τ_2, E).

1.7 Theorem

If (X, τ_1, τ_2, E) is a soft bitopological space then $\tau = \tau_1 \cap \tau_2$ is a soft topological space over X.

Proof

i). φ, X belong to $\tau_1 \cap \tau_2 = \tau$

ii). Let $\{(F_i, E) : i \in I\}$ be a family of soft sets in $\tau_1 \cap \tau_2 = \tau$. Then $\{(F_i, E) \in \tau_1\}$ and $\{(F_i, E) \in \tau_2\}$ for all $i \in I$.

Therefore $\bigcup_{i\in I}(F_i, E) \in \tau_1$ and $\bigcup_{i\in I}(F_i, E) \in \tau_2$.

Thus $\bigcup_{i\in I}(F_i, E) \in \tau_1 \cap \tau_2 = \tau$.

iii). Let $(F, E), (G, E) \in \tau_1 \cap \tau_2 = \tau$.

Then $(F, E), (G, E) \in \tau_1$ and $(F, E), (G, E) \in \tau_2$.

Since $(F, E) \cap (G, E) \in \tau_1$ and $(F, E) \cap (G, E) \in \tau_2$.

Therefore $(F, E) \cap (G, E) \in \tau_1 \cap \tau_2 = \tau$.

Thus $\tau_1 \cap \tau_2 = \tau$ defines a soft topology on X.

Hence, the (X, τ_1, τ_2, E) is a soft bitopological space then $\tau_1 \cup \tau_2$ is not a soft topological space over X.

$H(e_1) = F_2(e_1) \cup G_2(e_1) = \{h_2\} \cup \varphi = \{h_2\}$ and $H(e_2) = F_2(e_2) \cup G_2(e_2) = \varphi \cup \{h_2, h_3\} = \{h_2, h_3\}$ but $(H, E) \notin \tau$. Thus, τ is not a soft topology on X.

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