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Fractional Fourier-Wavelet Transform and Its Convolution Structure

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Abstract

This paper is concerned with the generalization of fractional Fourier-Wavelet transform in a distributional sense. In this paper, we discuss about product theorem, and convolution theorem for the Fractional Fourier-wavelet Transform.

Keywords: Fractional Fourier transform, Continuous Wavelet Transform, Test function space, Signal Processing, Convolution.

1. Introduction:

The Fractional Fourier transform is generalization of the Fourier transform which was firstly used in the field of signal processing by Victor Namias in 1980s[1]. The fractional Fourier transform has applications in areas of optics and signals processing and it is also a generalization of space (or time) and frequency domain which are central concepts of signal processing [2-4].

The Fractional Fourier transform of $f(x)$ is given by

$$FrFT\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, K_\alpha(x,p)\, dx \tag{1.1}$$

where, $K_\alpha(x,p) = C_\alpha e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha - 2xp]}$

where $C_\alpha = \sqrt{\frac{1-icot\alpha}{2\pi}}$

The Wavelet transform better tool for the analysis of non-stationary signals because it provides better time-frequency localization properties. The wavelet transform remove the non-stationary properties from the involved signals[5]. The continuous wavelet transform give simultaneous, high-resolution time and frequency information about a signal[6]. For the analysis of non-stationary signals and images the continuous wavelet transform is more effective because of its scale-space representation[7].

The continuous Wavelet transform of $f(t)$ is given by

$$W_\psi\{f(t)\}(b,a) = \int_{-\infty}^{\infty} f(t)\bar{\psi}_{b,a}(t)dt \quad , \ b \in R, a \in R-\{0\} \tag{1.2}$$

where $\psi_{b,a}(t) = \frac{1}{|a|} e^{-i\pi\left(\frac{t-b}{a}\right)^2}$.

The Fractional Fourier-Wavelet transform of $f(x,t)$ is defined as

$$FrFWT\{f(x,t)\} = F_\alpha(p,b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,t)K_\alpha(x,p,t,a,b)dxdt \tag{1.3}$$

where $K_\alpha(x,p,t,a,b) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]}\frac{1}{\sqrt{|a|}}e^{i\pi\left(\frac{t-b}{a}\right)^2}$

where $0<\alpha<\frac{\pi}{2}$.

$$K_\alpha(x,p,t,a,b) = Ce^{\frac{i}{2}[C_1(x^2+p^2)-2C_2xp]}e^{iC_3(t-b)^2}$$

where $C = \sqrt{\frac{1-icot\alpha}{2|a|\pi}}$, $C_1 = cot\alpha$, $C_2 = cosec\alpha$, $C_3 = \frac{\pi}{a^2}$

$K_\alpha(x,p,t,a,b)$ belongs to the testing function space and $f(x,\text{t})$ lies in its dual space.

The testing function space of the fractional Fourier-Wavelet transform is collection of an infinitely differentiable complex valued smooth function $\emptyset(x,t)$ on R^n belongs to $E(R^n)$, if for each compact set I $\subset S_{c,d}$

where $S_{c,d} = \{x,t{:}\, x,t \in R^n, |x| \le c,\ |t| \le d,\ c>0,\ d>0\}$

$$\gamma_{E,l,n}(\emptyset) = \sup_{\substack{x\in I\\ t\in I}}|D_x^l D_t^n \emptyset(x,t)| < \infty \tag{1.4}$$

Thus, $E(R^n)$ will denotes the space of all $\emptyset \in E(R^n)$ with support contained in $S_{c,d}$. Moreover, we say that f is a fractional Fourier-Wavelet Transformable if it is a member of E^*,the dual space of E.

The Distributional Fractional Fourier-Wavelet transform of $f(x,t) \in E(R^n)$ defined by

$$FrFT\{f(x,t)\} = F_\alpha(p,b) = \langle f(x,t),\ K_\alpha(x,\ p,\ t,\ a,\ b)\rangle \tag{1.5}$$

where $K_\alpha(x,p,t,a,b) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i}{2sin\alpha}[(x^2+p^2)cos\alpha-2xp]}\frac{1}{\sqrt{|a|}}e^{i\pi\left(\frac{t-b}{a}\right)^2}$

where $0<\alpha<\frac{\pi}{2}$.

Thus $K_\alpha(x,p,t,a,b)$ belongs to the testing function space and $f(x,\text{t})$ lies in its dual space.

2. Some Importa0nt Definition for Convolution Theorem

Definition1: For any function $f(x,t)$, let us define the function $\tilde{f}(x,t)$ and $\tilde{\tilde{f}}(x,t)$ by

$$\tilde{f}(x,t) = f(x,t)e^{i\left(\frac{C_1}{2}x^2+C_3t^2\right)}$$

$$\tilde{\tilde{f}}(x,t) = f(x,t)e^{-i\left(\frac{C_1}{2}x^2+C_3t^2\right)}$$

Definition 2: For any two functions f and g, we define the convolution operation $'\star'$ by

$$h(x,t) = (f\star g) = Ce^{-i\left(\frac{C_1}{2}x^2+C_3t^2\right)}\left(\tilde{f} * \tilde{g}\right)(x,t)$$

where $C=\sqrt{\frac{1-icot\alpha}{2a\pi}}$ and $'*'$ is the convolution operation for the Fourier transform defined as

$(f*g)(x,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u,v)g(x-u,t-v)dudv.$

Now we define the product operation $'\otimes'$ by

$$\left(\widetilde{\widetilde{F_\alpha}}\otimes\widetilde{\widetilde{G_\alpha}}\right)(p,a,b)=e^{i\left(\frac{C_1}{2}p^2+C_3b^2\right)}\left(\widetilde{\widetilde{F_\alpha}}*\widetilde{\widetilde{G_\alpha}}\right)(p,a,b)$$

3.

3.1 Convolution Theorem

Statement:

Let $h(x)=(f\star g)(x,)$ and $F_\alpha, G_\alpha\ and\ H_\alpha$ denote the Fractional Fourier-Wavelet Transform of $f,g,and\ h$ respectively. Then

$$H_\alpha(p,a,b)=\ e^{-iC_1p^2}e^{-iC_3b^2}F_\alpha(p,a,b)\,G_\alpha(p,a,b)$$

Proof:

From the definition of the fractional Fourier-Wavelet Transform

$$H_\alpha(p,a,b)=C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left[C_1\left(x^2+p^2\right)-2C_2xp\right]}e^{iC_3(t-b)^2}h(x,t)dxdt$$

Where $C=\sqrt{\frac{1-icot\alpha}{2|a|\pi}}$, $C_1=cot\alpha\ C_2=cosec\alpha$, $C_3=\frac{\pi}{a^2}$

$$H_\alpha(p,a,b)=C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left[C_1\left(x^2+p^2\right)-2C_2xp\right]}e^{iC_3(t-b)^2}\left(Ce^{-i\left(\frac{C_1}{2}x^2+C_3t^2\right)}\left(\tilde{f}*\tilde{g}\right)(x,t)\right)dxdt$$

$$=C^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left[C_1\left(x^2+p^2\right)-2C_2xp\right]}e^{iC_3(t-b)^2}e^{-i\left(\frac{C_1}{2}x^2+C_3t^2\right)}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}(u,v)\tilde{g}(x-u,t-v)dudv\right)dxdt$$

$$=C^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left[C_1\left(x^2+p^2\right)-2C_2xp\right]}e^{iC_3(t-b)^2}e^{-i\left(\frac{C_1}{2}x^2+C_3t^2\right)}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u,v)e^{i\left(\frac{C_1}{2}u^2+C_3v^2\right)}g(x-u,t-v)e^{i\left(\frac{C_1}{2}(x-u)^2+C_3(t-v)^2\right)}dudv\right)dxdt$$

$$H_\alpha(p,a,b)=C^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left[C_1\left(x^2+p^2\right)-2C_2xp-C_1x^2+C_1u^2+C_1(x-u)^2\right]}e^{i\left[C_3(t-b)^2-C_3t^2+C_3v^2+C_3(t-v)^2\right]}f(u,v)g(x-u,t-v)dudvdxdt$$

$$=C^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left[C_1\left(x^2+2u^2-2ux+p^2\right)-2C_2xp\right]}\,e^{i\left[C_3\left(t^2+2v^2+b^2-2tb-2vt\right]\right.}f(u,v)g(x-u,t-v)dudvdxdt$$

put $x-u=y\rightarrow dx=dy\ and\ t-v=z\rightarrow dt=dz$

$$= C^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left[C_1\left((u+y)^2+2u^2-2u(u+y)+p^2\right)-2C_2p(u+y)\right]}$$

$$e^{i\left[C_3\left((v+z)^2+2v^2+b^2-2b(v+z)-2v(v+z)\right)\right]} f(u,v)g(y,z)dudvdydz$$

$$= C^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left[C_1\left(u^2+y^2+p^2\right)-2C_2pu-C_2py\right)]} e^{i\left[C_3\left(v^2+z^2+b^2-2bv-2bz\right)\right]}$$

$$f(u,v)g(y,z)dudvdydz$$

$$= C^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left[C_1\left(u^2+p^2+y^2+p^2\right)-C_1p^2-2C_2pu-2C_2py\right)]}$$

$$e^{i\left[C_3\left(v^2-2bv+b^2+z^2-2bz+b^2-b^2\right)\right]} f(u,v)g(y,z)dudvdydz$$

$$= C^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left[C_1\left(u^2+p^2\right)-2C_2pu\right)]} e^{\frac{i}{2}\left[C_1\left(y^2+p^2\right)-2C_2py\right)]}$$

$$e^{iC_3(v-b)^2} e^{iC_3(z-b)^2} e^{-iC_3b^2} e^{-iC_1p^2} f(u,v)g(y,z)dudvdydz$$

$$H_\alpha(p,a,b) = C^2 e^{-iC_1p^2} e^{-iC_3b^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left[C_1\left(u^2+p^2\right)-2C_2pu\right)]} e^{iC_3(v-b)^2} f(u,v)dudv$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left[C_1\left(y^2+p^2\right)-2C_2py\right)]} e^{iC_3(z-b)^2} f(y,z)dydz$$

$$H_\alpha(p,a,b) = e^{-iC_1p^2} e^{-iC_3b^2} F_\alpha(p,a,b), G_\alpha(p,a,b)$$

3.2 Product Theorem

$$\left(\widetilde{\widetilde{F_\alpha}} \otimes \widetilde{\widetilde{G_\alpha}}\right)(p,a,b) = e^{i\left(\frac{C_1}{2}p^2+C_3b^2\right)} \left(\widetilde{\widetilde{F_\alpha}} * \widetilde{\widetilde{G_\alpha}}\right)(p,a,b)$$

$$= e^{i\left(\frac{C_1}{2}p^2+C_3b^2\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{\widetilde{F_\alpha}}(x,a,t)\, \widetilde{\widetilde{G_\alpha}}\,(p-x,a,b-t)dxdt$$

$$= e^{i\left(\frac{C_1}{2}p^2+C_3b^2\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\left(\frac{C_1}{2}x^2+C_3t^2\right)} F_\alpha(x,a,t)\; e^{-i\left(\frac{C_1}{2}(p-x)^2+C_3(b-t)^2\right)}\, G_\alpha(p-x,a,b-t)dxdt$$

$$= e^{i\left(\frac{C_1}{2}p^2+C_3b^2\right)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\left(\frac{C_1}{2}x^2+C_3t^2\right)} \left[C \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left[C_1\left(z^2+x^2\right)-2C_2xz\right]} e^{iC_3(w-t)^2} f(z,w)dzdw\right]$$

$$e^{-i\left(\frac{C_1}{2}(p-x)^2+C_3(b-t)^2\right)}\, G_\alpha(p-x,a,b-t)dxdt$$

$$= e^{i\left(\frac{C_1}{2}p^2+C_3b^2\right)}C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z,w)e^{-i\left(\frac{C_1}{2}x^2+C_3t^2\right)}e^{\frac{i}{2}\left[C_1\left(z^2+x^2\right)-2C_2xz\right]}e^{iC_3\left(w^2-2wt+t^2\right)}$$

$$e^{-i\left(\frac{C_1}{2}(p-x)^2+C_3(b-t)^2\right)}\,G_\alpha(p-x,a,b-t)dzdwdxdt$$

$$= e^{i\left(\frac{C_1}{2}p^2+C_3b^2\right)}C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z,w)e^{\frac{i}{2}\left[C_1\left(z^2+z^2-z^2-(p-x)^2\right)-2C_2xz\right]}e^{iC_3\left(w^2+w^2-w^2-2wt-(b-t)^2\right)}$$

$$G_\alpha(p-x,a,b-t)dzdwdxdt$$

$$= e^{i\left(\frac{C_1}{2}p^2+C_3b^2\right)}C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z,w)e^{iC_1z^2}\,e^{\frac{i}{2}\left[C_1\left(-z^2-(p-x)^2\right)-2C_2xz\right]}e^{2iC_3w^2}\,e^{iC_3\left(-w^2-2wt-(b-t)^2\right)}$$

$$G_\alpha(p-x,a,b-t)dzdwdxdt$$

Now put $p-x=u \rightarrow dx=-du \rightarrow x=p-u$

$$b-t=v \rightarrow dt=-dv \rightarrow t=b-v$$

$\left(\widetilde{\widetilde{F_\alpha}}\otimes\widetilde{\widetilde{G_\alpha}}\right)(p,a,b)$

$$= e^{i\left(\frac{C_1}{2}p^2+C_3b^2\right)}C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z,w)e^{iC_1z^2}\,e^{-\frac{i}{2}\left[C_1\left(z^2+u^2\right)+2C_2z(p-u)\right]}e^{2iC_3w^2}\,e^{-iC_3\left(w^2+2w(b-v)+v^2\right)}$$

$$G_\alpha(u,a,v)dzdwdudv$$

$$= e^{i\left(\frac{C_1}{2}p^2+C_3b^2\right)}C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z,w)e^{iC_1z^2-2C_2zp}\,e^{-\frac{i}{2}\left[C_1\left(z^2+u^2\right)-2C_2zu\right]}e^{2iC_3\left(w^2-wb\right)}\,e^{-iC_3\left(w^2-2wv+v^2\right)}$$

$$G_\alpha(u,a,v)dzdwdudv$$

$$= e^{i\left(\frac{C_1}{2}p^2+C_3b^2\right)}C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z,w)e^{iC_1z^2-2C_2zp}\,e^{-\frac{i}{2}\left[C_1\left(z^2+u^2\right)-2C_2zu\right]}e^{2iC_3\left(w^2-wb\right)}\,e^{-iC_3(w-v)^2}$$

$$G_\alpha(u,a,v)dzdwdudv$$

$$= e^{i\left(\frac{C_1}{2}p^2+C_3b^2\right)}C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(z,w)e^{iC_1z^2-2C_2zp}e^{2iC_3\left(w^2-wb\right)}\,g(z,w)(8\pi Ca^2\,sin\alpha)\,dzdw$$

$$\left(\widetilde{\widetilde{F_\alpha}}\otimes\widetilde{\widetilde{G_\alpha}}\right)(p,a,b) = e^{i\left(\frac{C_1}{2}p^2+C_3b^2\right)}(8C^2\pi a^2\,sin\alpha)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iC_1z^2-2C_2zp}e^{2iC_3\left(w^2-wb\right)}\,f(z,w)g(z,w)\,dzdw$$

4. Conclusion

In this paper Convolution theorem and Product Theorem for Fractional Fourier-Wavelet transform is proved.

References

1. H M Ozaktas, Z Zalevsky, M A Kutay, “The Fractional Fourier transform with applications in Optics and Signal Processing”, John Wiley, Chichester, New York, USA, 2001.
2. E Sejdic, I Djurovic, L Stankovic, “Fractional Fourier transform as a signal processing tool: An overview of recent developments”, Signal Processing 91 (2011) 1351–1369.
3. H M Ozaktas, D Mendlovic, “Fourier fransform of fractional order and their optical Interpretation”, Optics Communication, vol. 101, pp 163-169, 1993
4. L M Bernardo, O D D Soares, “Fractional Fourier transform and optical system”, Optics Communication, vol. 110, pp 517-522, 1994
5. O Rioul, M Vetterli, “Wavelets and signal processing”, IEEE Signal Process, Mag. (October 1991) 14-38
6. C E Heil, D F Walnut, Continuous and Discrete Wavelet Transform, SIAM Rev. 31(December 1989) 730-736
7. F Antonio, P Rendón, R Robles, The convolution theorem for the continuous wavelet tranform, Signal Processing, Volume 84, Issue 1, 2004, Pages 55-67, ISSN 0165-1684
8. V. Namias, “The fractional order Fourier transform and its application to quantum mechanics,” J. Inst. Maths. Applics., vol. 25, pp. 241-265, 1980.
9. A Zayed “A convolution and product theorem for the fractional fourier transform”, IEEE Signal Processing Letters, 5(4):101–103, April 1998.
10. V D Sharma, P B Deshmukh, “A Convolution theorem for the Two-Dimensional Fractional Fourier Transform in Generalized sense”, 3rd International Conference on Emerging Trends in Engineering and Technology, ICETET 2010