



In polaron problem Path integrals for two time actions

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Abstract

The two-time action functional are used very often to describe the evolution of physical systems in random environment. The path integral method is most suitable to handle these problems as there is no underlying Hamiltonian (Lagrangian) which can describe the system.

In this paper we discuss in detail the evaluation of a functional integral associated with a non local two-time action functional. First we derive an exact expression for the path integral associated with a particular two-time quadratic action functional. It is then indicated how this result can be generalized to included general quadratic action functional. The result for the propagator for the general case is quoted. We then proceed to discuss the first cumulant approximation method to evaluate the path integral associated with non-quadratic two-time action functional and illustrate it by the particular example of the evaluation of density matrix expressed as a path integral for the polaron problem. This expression is used to obtain the ground state energy and the effective mass of polaron. It is shown that estimates for the ground state energy agrees with the earlier result. However, effective mass evaluated by using the Feynman definition diverges in the limit as the absolute temperature goes to zero. Thus this definition can not be used to describe polaron at finite temperature. In this context, an alternative definition due to Saitoh has been examined and found to be consistent. It is argued that the Feynman definition leads to a divergence in effective mass at zero temperature limit as it has no precise operational meaning.

KEYWORDS :- (PROPOGATOR, TIME ACTION, CUMULANT APPROXYMATION, POLARON, NON-LOCAL)

1. Introduction

In the context of path-integrals, non-local actions appear on several occasions¹. It was Feynman², who for the first time introduced these type of actions when he expressed the density matrix of a "Polaron" as a functional local action. Since then these action have been used to study several problem involving the motion of a system in random environment.

In order to see the origin of these actions in the context of path-integrals transparently, let us consider two system X and Y characterized by Lagrangians L_X and L_Y respectively. The total Lagrangian L associated with the coupled system $[X, Y]$ can be represented as:

$$L = L_X + L_Y + L_1 \dots\dots\dots (1)$$

Where, L_Y demptes the omteraction Lagrangian. The quantum dynamics of this system will be governed by the propagator $K(x, y'', x', y', T)$, which can be expressed as a path-integral.

$$K(x'', y'', x', y', T) = \int \exp(iS/\hbar) D[x(t)]D[y(t)] \dots\dots\dots (2)$$

where, x and y denote respectively the coordinates associated with systems X and Y . S denotes the classical action-functional and is defined as:

$$S = \int_0^T L dt \dots\dots\dots (2a)$$

In many situations of practical interest one is usually interested in the properties one of the sub systems (say Y) alone. These properties are expressed as some average over the configurations of X . The system X acts typically as a reservoir and in most of the cases it is modeled as a collection of harmonic oscillators (HO). Thus L_x is quadratic in coordinates and velocities. Next, lest us also assume that the interaction Lagrangian is of the form $x, f(y)$. Hence for this model S can be written as:

$$S = \int_0^T (m/2)(\dot{X}^2 - \omega^2 X^2) dt + S_Y + \int_0^T x \cdot f(y) dt \dots\dots\dots (3)$$

Since, S is quadratic in x , the path-integration over the coordinate x can be performed using the result derived by Feynman³. The propagator K , as a result path-integration over x , takes the form,

$$K(x'', y''; x', y', T) = K_{HO}(X''; X'.T) \int \exp(iS_{eff}(y)/\hbar) D[y(t)] \dots\dots\dots (4)$$

where $K_{HO}(X''; X'.T)$ denotes the usual expression for the propagator associated with an harmonic oscillator of frequency ω . S_{eff} is an effective action functional involving the coordinates of Y alone. The general structure of S_{eff} can be easily derived and reads as

$$S_{eff} = S_Y + a(x'', x'). \int_0^T f(y) dt + \int_0^T dt \int_0^T ds G(t, s) f(y(t)) \cdot f(y(s)) \dots\dots\dots (5)$$

where, G and a are some functions of their arguments. Note that S_{eff} is now non-local in coordinate y . thus the non-locality is a result of partial path-integration. Further, K of eq. (4) still involves the information about system X through it's end-points. In many cases involving fadom environment, we are usually interested in the average properties of system Y . the average propagator governing the evolution of Y is obtained by averaging K of eq. (4) over all configurations of X . the evaluation of average propagator involves path-integration of an S_{eff} which is similar to the S_{eff} of eq. (5) but involves the coordinate of Y alone.

In this paper we would consider the problem of evaluating the path -integral associated with the non-local actions of type (5). The evaluation of such path-integral in closed analytical form is possible only in very few cases. In all other cases the path-integral is usually evaluated in the so called first cumulant approximation. This method uses a known expression for the path-integral as a first guess to the actual solution which is subsequently improved upon. The guess is usually chosen to correspond to propagator associated with a ‘non-local’ quadratic action functional of the form

$$S_T = \int_0^T dt -\frac{x^2}{2} + \int_0^T dt \int_0^T ds G(t, s)[x(t) - x(s)]^2 \dots\dots\dots (6)$$

The first guess for the path-integral is usually referred to as trial path-integral, and the corresponding action as the trial action. The action functional S_T of eq. (6) is chosen such that the corresponding propagator is known in closed analytical form. The propagators associated with non-local actions described by eq. (6) have been recently evaluated in closed analytical form⁵. Thus we organise the paper as follows.

In the next section we shall explicitly evaluate the propagator associated with S_T of eq. (6). To keep the discussion at an elementary level we shall make further simplifying assumption and only consider the case when the kernel $G(t, s) = \text{Constant}$. We shall then consider the case when $G(t, s)$ happens to be an arbitrary function of its argument and indicate how the method employed to treat the simplified problem can also be used to solve this general case. We will conclude this section by giving explicit form of propagator for the cases when $G(t, s)$ is an arbitrary function of its arguments.

In section 3, we shall consider the evaluation of propagator for general non-local action functional within the first cumulant approximation. Again we shall illustrate the method through a particular example rather than the general case. Since this conference is devoted to the memory of late Richard P. Feynman, we have, as a tribute to this great master, selected the example of the classic Polaron problem to illustrate the method. Finally we use this approximate expression for “Polaron Propagator” to derive the estimates for the ground state energy and effective mass of Polaron.

2. Evaluation of the Propagator for the Non-Local Quadratic Actions

In this section we shall first evaluate the propagator for the particle characterized by the non-local action functional:

$$S = (m/2) \int_0^T dt x^2 - (m\Omega^2/4T) \int_0^T dt \int_0^T ds [x(t) - x(s)]^2 \dots\dots\dots (7)$$

where m is mass. T is the *total* time of evaluation and Ω characterizes the interaction strength. The action functional of eq. (7) has been used as a model to study the spectrum of an electron in disordered solids as early as 1969 by Bezak⁶. Subsequently it has been extensively used as trial action to evaluate the propagator associated with more complicated action functionals. However, the exact path integration of eq. (7) was first carried out by Samathiakanit⁷ and later by several authors⁸⁻¹¹ using different technique of path-integrations. As has been pointed out earlier that S of eq. (7) results when further simplifying assumption on the nature of $G(t, s)$ in eq. (6) is made.

In this paper we shall, however, follow a different method to evaluate the propagator associated with S of eq. (7). The method as applied to the simple case of action in eq. (7) is due to Papadopoulos⁸ and can be generalized to the general case providing an alternative method to that of Ref. 5 for the evaluation of propagator for the general non-local quadratic actions. In this paper, however, we shall merely outline the steps for evaluating the propagator for S of eq. (6) and discuss in detail the evaluation of propagator associated with S of eq. (7). The propagator associated with S of eq. (7) can be written as

$$K(X'', x, T) = \int \exp[iS/\hbar] D[x(t)] \dots\dots\dots (8)$$

Which can be re-expressed as:

$$K(X''; x', T) = \int \exp(iS_0/\hbar) \exp\left[im\Omega^2 \left(\int_0^T x dt\right) / 2\hbar T\right] D[x(t)] \dots\dots\dots (9)$$

where the action functional S_0 has the following form:

$$S_0 = (m/2) \int_0^T (\dot{x}^2 - \Omega^2 x^2) dt \dots\dots\dots (10)$$

Next we make use of the following Gaussian identity

$$(\pi m / i\hbar T)^{3/2} \exp\left[im\Omega^2 \left(\int_0^T x dt\right)^2 / 2\hbar T\right] = \int_{-\infty}^{\infty} du \exp\left[-\frac{i\hbar T u^2}{2m} + i\Omega u \int_0^T x dt\right] \dots\dots\dots (11)$$

and write K in eq. (9) as

$$K(X; x', T) = (i\hbar T / \pi m)^{3/2} \int_{-\infty}^{\infty} du \exp[-i\hbar T u^2 / 2m] K_1(u) \dots\dots\dots (12)$$

where,

$$K_L(u) = \int D[x(t)] \exp[iS_L/\hbar] \dots\dots\dots (13a)$$

is the propagator associated with the action functional S_L :

$$S_L = \int_0^T dt [(m/2)(\dot{x}^2 - \Omega^2 x^2) + \Omega u \cdot x/2] \dots\dots\dots (13b)$$

Next, we observe that S_L is identical to the action functional for an harmonic oscillator of frequency Ω , perturbed by a constant force $\Omega u/2$. The expression for the propagator K_1 associated with action functional S_L is well known and can be found explicitly in Ref. (3). The evaluation of K_1 has also been discussed in detail by Lawande⁴ in his contribution to this conference. We substitute the expression for K_1 from Ref. (3) and carry out the remaining integration over u to arrive at expression for K in closed form. This expression reads as:

$$K(X'', X', T) = (m/2\pi i\hbar)^{3/2} (\Omega T / 2 \sin(\Omega T / 2))^3 \exp\left[im \cot(\Omega T / 2) (x'' - x')^2 / 4\hbar\right] \dots\dots\dots (14)$$

The central point in the evaluation of the propagator, was the use of identity (11) which enabled us to express the non-local term as a Gaussian average of an exponent involving *local potential terms*. Such a procedure can be applied to treat the problem of evaluating the propagator associated with the general non-

local quadratic action involving arbitrary $G(t, s)$. To this end, first note that $G(t, s)$ being a symmetric function of its arguments admits an expansion

$$G(t, s) = \sum_n \mu_n \psi_n(t) \psi_n(s) \dots\dots\dots (15)$$

where ψ_n are a complete set of ortho-normal eigenfunctions of the integral equation

$$\int_0^T G(t, s) \psi_n(s) ds = \mu_n \psi_n(t) \dots\dots\dots (16)$$

Therefore the term

$$I = \int_0^T dt \int_0^T G(t, s) [x(t) - x(s)]^2 \dots\dots\dots (17)$$

Hence we can use a multidimensional analogue of the Guassian identity (11) to replace the exponent of the second term in eq. (18) as the Guassian average of a local potential while evaluating the propagator. Thereafter the steps similar to those followed in eq. (12-14) will yield the expression for the propagator for the general case. In what follows, however, we shall merely quote the resulting expression for the propagator. The expression for the propagator associated with general non-local quadratic action functional eq. (6) has the form:

$$K = \left(\frac{m}{2\pi i \hbar Q_1 Q_2} \right)^{1/2} \exp[iS_d/\hbar] \dots\dots\dots (20)$$

Where the quantity Q_1 is given by

$$Q_1 = \left(\int_0^T dt \int_{-1}^0 du R(t, u) \right) \dots\dots\dots (21)$$

$R \sim mg$ the solution of the integro-differential equation

$$A(\mu, t, s)R(t, s) = G(t, S) \dots\dots\dots (22a)$$

along with the condition $R(0, s) = R(T, s) = 0$. The integro-differential operator $A(\mu, t, s)R(t, s)$ is defined as:

$$A(\mu, t, s)f(t) = \frac{m}{4} \frac{\delta^2}{\delta t^2} f(t) + \Gamma(t)f(t) + \mu \int_0^T ds G(t, s)f(s)$$

With

$$\Gamma(t) = \int_0^t ds G(t, s) \dots\dots\dots (22b)$$

Next, the quantity Q_2 has the expression

$$Q_2 = \frac{\eta(T)\xi(0) - \xi(T)\eta(0)}{\eta(0)\xi(T) - \xi(0)\eta(T)} \dots\dots\dots (23)$$

ξ, η being the two linearly independent solutions of equation

$$A(0, t, s): f = 0 \dots\dots\dots (24)$$

Lastly, the quantity S_d , the action functional along the classical trajectory which is the solution of equation

$$A(l, t, s)f(t) = 0, f(0) = x', f(T) = x'' \dots\dots\dots (25)$$

can be expressed as

$$S_d = m(x'' - x')^2 / 2(v(T) - v(0)) \dots\dots\dots (26)$$

$v(t)$ being the non-constant solution of eq. (25). This completes the task of writing down *the* expression for *the* propagator for the action functional of eq. (6). We shall now proceed with the discussion of the Polaron problem and thereby illustrate *the* approximation scheme for evaluation of propa gators associated with general non-local action functionals.

3. The Polaron Problem

The polamn problem has continued to attract the attention of physicists since the time it was introduced by Landau¹² and subsequently developed by Frohlich.¹³ Briefly a polaron is an electron moving in a polar crystal together with self induced polarization of lattice. As a consequence of electron-phonon interaction the polaron tends *to* have a lower ground state energy and higher effective mass as compared to that of a hare electron. These two important parameters were first obtained by Frohlich¹³ for small electron-phonon couplings using a perturbation approach. However, the method could not be used for arbitrary strength of electron-phonons. It was this aspect which prompted Feynman² to use the path-integral method coupled to a variational scheme and he succeeded in obtaining an estimate for the ground state energy which remains the best to date. The estimate of effective mass very critically depends on the definition. Moreover, it has also not been possible to obtain bounds on the effective mass. One, therefore, is solely guided by the experiments. With these remarks we proceed to evaluate 'the polaron propagator.'

The Lagrangian for the joint electron-phonon system can be written as:

$$L = \frac{x^2}{2} + \sum_k \left(\frac{q_k^2}{2} + q_k^2 \right) + (\alpha\pi\sqrt{2})^{1/2} \sqrt{2q_k \exp[i K \cdot x_0] / k} \dots\dots\dots (27)$$

where x refers to the electron co-ordinate. The co-ordinate \mathbf{qk} denotes the k th phonon mode. For simplicity we have assumed that all phonons have the same frequency and we have scaled the variables in the units such that the Planck's constant \hbar , the mass of bare electron m and the phonon frequency ω_0 are all equal to one. Thus the first term in eq. (27) represents the Lagrangian for free electron, the second is the Lagrangian for free phonons while the last term represents the interaction between electrons and phonons. α denotes the interaction strength. The statistical mechanical properties of this system will be described by the two point density matrix $\rho(x'', x', q'_k; \beta)$ which can be expressed as a path-integral:¹⁴

$$\rho(x'', x', q'_k q_k; \beta) = \int \exp(-S) D[x(t)] D[q(t)] \dots\dots\dots (28)$$

with S , as usual is given by:

$$S = \int_0^\beta L dt \dots\dots\dots (29)$$

Next notice that S is quadratic in q_k and hence path-integration over q'_k is straight forward. When a partial path-integration over phonon coordinates is performed, we reduce the problem to that of path-integration of an effective action is involving election coordinates alone. The general structure of the effective action is similar to the action in eq. (7) where f is identified as $\mathbf{e}^{\mathbf{k}} \cdot \mathbf{x} / k$ and involves phonon end points. Now, phonon end-points are eliminated depending on the assumptions made to study the problem. Feynman assumed, that the low temperature properties of polaron are completely determined by the matrix element of the two point density matrix between the ground state of free phonons. This leads to⁴ one expression for the effective action functional. On the other hand, b) if the phonon end-points are eliminated by evaluating the trace of density matrix ($q''_k = q'_k$ for all k) over phonon states which is the correct procedure for evaluation

the partition function, it leads to another expression for the effective action. These expressions have the form:

$$S_{eff} = \frac{\int_0^\beta dt x^2}{2} - (\alpha\pi\sqrt{2}) \int_0^\beta dt \int_0^\beta ds \int dk G(t,s) \exp[iK.(x(t) - x(s))]/(2\pi)^3 \dots\dots\dots (30a)$$

Where

$$G(t,s) = \exp(-|t - s|) \text{ for case a } \dots\dots\dots (30b)$$

$$= ch[\beta/2 - |t - s|]/s^h \beta/2 \text{ for case b } \dots\dots\dots (30c)$$

Thus the reduced density matrix for the polaron is given by the path-integral

$$\rho(x'', x', \beta) = \int \exp[-S_{eff}] D[x(t)] \dots\dots\dots (31)$$

The path-integral in eq. (31), as has been pointed out earlier, can not be evaluated in closed analytical form and is scheme, as a first guess, S_{eff} is approximated by some function S_o , known as trial action fictional.

The choice of S_o is arbitrary to the extent that $\rho_o(x'', x', \beta)$, the density matrix associated with is exactly known, of course the ‘deviation’ of ρ from ρ_o will be less, if the ‘deviation’ of S_{eff} from S_o is ‘minimal’.

The path-integral expression for $\rho(x'', x', \beta)$ in eq. (31) can be re-expressed as:

$$\rho(x'', x', \beta) = \rho_o(x'', x', \beta) \langle \exp(S_{eff} - S_o) \rangle_{S_o} \dots\dots\dots (32)$$

where the symbol $\langle A \rangle_{S_o}$ stands for the average of A with respect to S_o and is defined as:

$$\langle A \rangle_{S_o} = \int D[x(t)] \exp(-S_o) A \rho_o \dots\dots\dots (33)$$

The first cumulant approximation consist of approximating eq. (32) as

$$\begin{aligned} \rho(x'', x', \beta) &= \rho_o(x'', x', \beta) \langle \exp[-(S - S_o)] \rangle_{S_o} \\ &= \rho_o(x'', x', \beta) \exp[-\langle (S - S_o) \rangle_{S_o}] \dots\dots\dots (34) \end{aligned}$$

Thus the task of obtaining ρ within first cumulant approximation is equivalent to that of obtaining $\langle S_{eff} - S_o \rangle_{S_o}$. We shall now proceed with the evaluation of $\langle S_{eff} - S_o \rangle_{S_o}$. However, we shall restrict ourselves to the case of small α for simplicity. In this case S_o can be chosen as the free particle action, viz.

$$S_o = 1/2 \int_0^\beta x^2 dt \dots\dots\dots (35a)$$

and the corresponding expression for $\rho_o(x'', x', \beta)$ is

$$\rho_o(x'', x', \beta) = (1/2\pi\beta)^{1/2} \exp[-1/2\beta(x'' - x')^2] \dots\dots\dots (35b)$$

With this expression for S_o we find

$$\begin{aligned} \langle S_{eff} - S_o \rangle_{S_o} &= -\alpha \pi\sqrt{2} \int_0^\beta dt \int_0^\beta ds \int d^3 k G(t,s) \langle \exp\{ik.(x(t) - x(s))\} \rangle_{S_o} / (2\pi)^3 \\ &= \frac{-\alpha\pi\sqrt{2}/\rho_o}{\rho_o} \int_0^\beta dt \int_0^\beta ds \int \frac{d^3 k}{(2\pi)^3} G(t,s) \int D[x(t)] \exp\left(-\int_0^\beta x^2 dt + ik.(x(t) - x(s))\right) \end{aligned} \quad (36)$$

Again we observe that the path-integral in eq. (36) corresponds to the propagator (with the identification of β as imaginary time) associated with a free particle perturbed by a constant force $\eta(\xi) = ik(\delta(\xi - t) - \delta(\xi - s))$. The expression for the propagator for this case is well known and can be found in Ref. (3). Substituting this expression for the propagator and carrying out some simplification we arrive at the expression for the density matrix of polaron.

$$\rho(x'', x', \beta) = \rho_o(x'', x', \beta) \exp[-\alpha\sqrt{2}/\pi] \int_0^\beta du (\beta - u) G(u) \int_0^\infty dk \exp[-k^2 b] \sin(ka)/ka \dots\dots (33)$$

Where

$$a = |x'' - x'| u/\beta$$

$$b = (\beta - u)u/2\beta \dots\dots\dots (38)$$

Valid for small values of α . When α is large S_o of eq. (35a) does not serve the purpose of a good trial action because the departure of S_{eff} from free particle behaviour is significant. In such a case one usually chooses a trial action similar to that of (6). The kernel $G(t, s)$ is chosen to have form as in eq. (30) and has some free parameters. For this case too the path-integral representing density matrix for the polaron in first cumulant approximation can be expressed as in eq. (38). Subsequent explicit evaluation of the path-integral has been carried out by us in Ref. (1). However, the calculation being very detailed, we shall not discuss it in this paper. With these remarks we return to the discussion of the physical properties of polaron for small values of α .

The free energy of polaron F is related to the density matrix by the relation

$$F = -\ln \text{Tr}(\rho(x'', x', \beta))/\beta$$

$$= -\ln[(0.0. \beta)]/\beta \dots\dots\dots (39)$$

apart from a volume factor.

Substituting the expression for ρ from eq. (37) in eq. (39), we obtain:

$$F = -3\ln(2\pi\beta)/2\beta - (\alpha/\pi\beta\sqrt{8}) \int_0^\beta dt \int_0^\beta ds G(t, s)/\sqrt{b} \dots\dots\dots (40)$$

$$\text{And hence the ground state energy } E_o = \lim_{\beta \rightarrow \infty} F \approx -\alpha + O(\alpha^2) \dots\dots\dots (41)$$

irrespective of the choice (30b) or (30c) for the kernel $G(t, s)$. This expression for the free energy coincides with the one derived earlier by Fröhlich¹³ and Feynman who considered only $G(t, s)$ as described by eq. ((30b).

Let us now investigate the other physical quantity namely the effective mass. Feynman² conjectured that in the limit $\beta \rightarrow \infty$, the polaron density matrix will approach the free particle density matrix. Therefore, he argued that as $\beta \rightarrow \infty$ and $|x'' - x'| \rightarrow 0$, ρ will have the form

$$\rho \approx \exp[-m^* |x'' - x'|^2/2\beta] + O(|x'' - x'|^4) \dots\dots\dots (42)$$

m^* being the effective mass. He then proceeded to evaluate m^* with $G(t, s)$ as given in eq. (30b). His result coincided with the earlier result of Fröhlich. They found that the choice (30b) for $G(t, s)$ does indeed leads to an expression for m^* which is in agreement with earlier result. However, as has been pointed out earlier, if a correct procedure, namely the tracing over the phonon end-points, is employed it would lead to an expression for S_{eff} where $G(t, s)$ is given by eq. (30c). When al evaluated p and subsequently m^* with this of $G(t, s)$, they found that Feynman conjecture regarding the asymptotic behaviour of polaron density matrix is not correct. If one still pursues the Feynman definition (42) and tries to evaluate the effective mass one arrives at a *divergent* expression for this parameter. To see this clearly we express ρ of eq. (38) in the form

$$\rho_0(x'', x', \beta) \exp[(Z_1 + Z_2 |x'' - x'|^2)/\alpha \pi\sqrt{2}] + O(|x'' - x'|^4) \dots\dots\dots (43)$$

Where,

$$Z_1 = \sqrt{\pi} \int_0^\beta du (\beta - u) G(u)/2\sqrt{b} \dots\dots\dots (44)$$

$$Z_1 = \sqrt{\pi} \int_0^\beta du (\beta - u) u^2 G(u) / 24\beta^2 b^{3/2} \dots\dots\dots (45)$$

$$m^* = 1 + (\alpha\sqrt{\beta}/3\sqrt{\pi}) \int_0^\beta du \{\beta - u\}^{1/2} G(u) \dots\dots\dots (46)$$

It can now be verified easily that if G is chosen as in eq. (30b), we get

$$m^* = 1 + \alpha/6 \dots\dots\dots (47)$$

Which is the familiar expression earlier derived by Frohlich¹³ and later by Feynman². However, when we substitute the expression for G as in eq. (30c) we obtain

$$m^* = 1 + \alpha\beta/3 + \alpha/12 \dots\dots\dots (48)$$

Which diverges as $\beta \rightarrow \infty$. Even the first non-divergent correction as $\alpha/12$ as against $\alpha/6$ in eq. (47). Apparently, there is no way of removing the divergent character and we conclude that the definition of m^* as given in eq. (42) is inadequate. This led us to examine the other prevailing definitions¹⁶⁻¹⁹ of effective mass in literature. We found that the discrepancy between the two expressions for m^* persists for other definitions as well except the one given by Saitoh¹⁷ and Peeters and Devreese. Saitoh considered the polaron under the influence of a vanishingly small force f and defined the effective mass in terms of the diagonal element of density matrix

as:

$$\rho(0,0,\beta) \approx \exp(-\beta^3 f^2 / 24m^*) \dots\dots\dots (49)$$

To employ this definition, we have to modify the definition of trial action (30a) to include the force term $\int_0^\beta x dt$. A straight forward calculation along the similar lines leads to the modified expression for polaron density matrix. This expression reads as:

$$\rho(0,0,\beta) = [1/(2\pi\beta)^{3/2}] \exp \left[(\alpha/\pi\sqrt{2}) \int_0^\beta dt \int_0^\beta ds G(t,s) \int_0^\infty \sin(k.f |A|) \exp(-k^2 b) / k.f |A| \right] \dots\dots\dots (50a)$$

Where,

$$A = (t - s)(\beta - t - s) / 2 \dots\dots\dots (50b)$$

The expression for eq. (50a) can be expanded in powers of f and we obtain

$$\rho(0,0,\beta) = [1\sqrt{(2\alpha\beta)}] \exp [(\alpha\sqrt{2}/\pi\sqrt{m})\{Z_1 + f^2 Z'_2 + \dots\}] \dots\dots\dots (51)$$

Where

$$Z'_2 = (\sqrt{\pi}/288\beta^2) \int_0^\beta du (\beta - u)^3 G(u) / b^{3/2} \dots\dots\dots (52)$$

This leads to an effective mass formula

$$\frac{1}{m^*} = 1 - \left(\alpha / 3\sqrt{\pi\beta^3} \right) \int_0^\beta du G(u) \sqrt{[u(\beta - u)^3]} \approx 1 - \alpha/6 + O(1/\beta) \dots\dots\dots (53)$$

irrespective of the choice (30b) or (30c) for the function $G(u)$.

Thus we see that the effective mass defined by Saitoh yields consistent results with regards to choices (30b) or (30c) for $G(t, s)$. Another definition which yields consistent results for the effective mass with respect to choices for G is given by Peeters and Devreese. This is perhaps related to the fact that both of these

definitions are directly based on the response of polaron under a small perturbative force. In contrast, the other definitions of the effective mass have no precise operational meaning.

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