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Certain Integral representation of Appell functions F_1, F_2 and F_3

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Abstract -- In this paper the authors define and investigate integral representations of Appell Functions F_1, F_2 and F_3 which is positive definite matrices and real symmetric. The results deal with real valued scalar functions of matrix argument in real case with matrices order unity. The objective of this paper is to determine special integral representations of Appell functions F_1, F_2 and F_3 of two matrix arguments.

Keywords: Multiple Hypergeometric functions; Appell function, Integral Representation, Hypergeometric functions, Special Functions.

1. INTRODUCTION

Definition of Appell functions F_1, F_2 and F_3 for matrix arguments is used to derive some integral representation of these Appell functions of matrix argument by using same approach as by Herz[2], Hua[3] and Mathai[4]. These results are useful in deriving properties of Appell function. In a similar manner new integral representations are obtained for lauricella functions and consequently for Appells functions F_1, F_2 and F_3 by Mathai[5]. From these derivations of Mathai it is clear that the method does not provide representation for F_4.

There are number of cases where Appell series are defined and obtained summation formula and integral representation of these functions in the literature, see for example Exton (1976), Karlson (1976), Srivastava (1981).

2. DEFINITIONS AND PRELIMINARIES

The Appell hypergeometric series arises frequently in various physical and chemical applications. It is defined by

$$F_1(\alpha,\beta,\beta';\gamma;\ x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta'_n)}{(\gamma)_{m+n}}\frac{x^m y^n}{m!n!} \tag{2.1}$$

$$\max\{|x|,|y|\}<1;$$

$$F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta'_n)\;x^m y^n}{(\gamma)_m\;(\gamma')_n\quad m\,!n\,!} \tag{2.2}$$

$$|x| + |y| < 1;$$

$$F_3(\alpha,\alpha',\beta,\beta';\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta'_n)\;x^m y^n}{(\gamma)_{m+n}\quad m\,!n\,!} \tag{2.3}$$

$$\max\{|x|,|y|\} < 1;$$

The following notations will be employed throughout this paper. A prime will denote a transpose of a matrix. All the matrices are assumed to be m × m real symmetric positive definite unless otherwise stated. Notation for A > 0 represents A as positive definite matrix and A > B represents A – B as positive definite matrix. R(·) means the real $\int_0^1(\;)$I – A > 0, that is, all the eigenvalues of A are between zero and 1. The notation || stands for norm of (·). The matrices are symmetric positive definite therefore the largest eigenvalues can be taken as the norms. (·) will denote determinant of (·) Tr(·) represents trace of (·) means the sum of leading diagonal elements of (·). Γ_m (·) stands for the generalized gamma function defined by

$$\Gamma_n(\alpha) = \Pi^{n(n-1)/4}\Gamma(\alpha)\Gamma\left(a-\frac{1}{2}\right)\ldots\ldots\ldots\;\Gamma\left(a-\frac{n-1}{2}\right) \tag{2.4}$$

where Re $(a) > \frac{n-1}{2}$.

3. APPELL FUNCTION F_1 OF MATRIX ARGUMENTS

The Appell function F_1 of matrix arguments represented by F_1 (α, β, β'; γ; X, Y) comparable to the corresponding scalar case and defined by the integral

$$F_1(\alpha,\beta,\beta';\gamma;X,Y)=\frac{\Gamma_n(\gamma)}{\Gamma_n(\alpha)\,\Gamma_n(\gamma-\alpha)}\int_0^1 |U|^{\alpha-(n+1)/2}\times|I-U|^{\gamma-\alpha-(n+1)/2}\,|I-UX|^{-\beta}\,|I-UY|^{-\beta'}\,dU \tag{3.1}$$

for Re $(\gamma) > \frac{n-1}{2}$, Re $(\alpha) > \frac{n-1}{2}$, Re $(\gamma-\alpha) > \frac{n-1}{2}$, X = X' > 0, Y = Y' > 0, || X || < 1 and || Y || < 1.

Integral representation of Appell function F_1 of matrix arguments are considered in the following theorems.

Theorem 1 –

$$F_1(\alpha,\beta,\beta';\gamma;X,Y)=\frac{\Gamma_m(\delta)\,\Gamma_m(\delta')}{\Gamma_m(\beta)\,\Gamma_m(\beta')\,\Gamma_m(\delta-\beta)\,\Gamma_m(\delta'-\beta')}$$

$$\times\int_0^I\int_0^I |U|^{\beta-(m+1)/2}|V|^{\beta'-(m+1)/2}|I-U|^{\delta-\beta-(m+1)/2}$$

$$\times|I-V|^{\delta'-\beta'-(m+1)/2}\,F_1\left(\alpha,\;\delta,\;\delta'\,;\;\gamma\,;X^{1/2}\,U\,X^{1/2},\;Y^{1/2}\,V\,Y^{1/2}\right)dU\,dV. \qquad \ldots(3.2)$$

where Re $(\beta) > \frac{m-1}{2}$, Re $(\beta') > \frac{m-1}{2}$, Re $(\delta) > \frac{m-1}{2}$, $X = X' > 0$, $Y = Y' > 0$, || X || < 1, || Y || < 1.

Proof: If we apply (3.1) on the right-hand side of (3.2), and change the order of integration, that is possible for the reason that the given integral is absolute convergent involved in the process and integrate with the help of the integral

$$ {}_2F_1(\alpha, \beta; \gamma; Z^{1/2} X Z^{1/2}) = \frac{\Gamma_m(\gamma)}{\Gamma_m(\alpha)\ \Gamma_m(\gamma-\alpha)} \int_0^I |U|^{\alpha-(n+1)/2} \times |I - U|^{\gamma-\alpha-(m+1)/2} |I - U^{1/2} Z^{1/2} XZ^{1/2} U^{1/2}|^{-\beta} dU, \qquad ...(3.3) $$

where Re $(\alpha) > \left(\frac{m-1}{2}\right)$, Re $(\gamma) > \left(\frac{m-1}{2}\right)$, Re $(\gamma - \alpha) > \left(\frac{m-1}{2}\right)$ and $\| XZ \| < 1$,

Further we have,

$$ {}_2F_1(\alpha, \beta; \beta; Z) = {}_1F_0(a; Z) = |I - Z|^{-\alpha} \quad \text{for } \| Z \| < 1. \qquad ...(3.4) $$

The result (3.2) follows subject to the following conditions:

$\| X^{1/2} U X^{1/2} \| \le \| X \| . \| U \| \le \| X \|$, since $\| U \| < 1$

and $\| Y^{1/2} V Y^{1/2} \| \le \| V \| . \| Y \| \le \| Y \|$, since $\| V \| < 1$.

Theorem 2 can be proved in the similar manner.

Theorem 2 –

$$ \Gamma_m(\beta)\,\Gamma_m(\beta')\,F_1(\alpha, \beta, \beta'; \gamma; X, Y) = \int\limits_{R>0} \int\limits_{S>0} \exp(-\mathrm{tr}(R + S))\,|R|^{\beta-(m+1)/2}\,|S|^{\beta'-(m+1)/2} $$

$$ \times\ {}_1F_1(\alpha; \gamma; X^{1/2} RX^{1/2} + Y^{1/2} SY^{1/2})\, dR\, dS. \qquad ...(3.5) $$

where Re $(\beta) > \left(\frac{m-1}{2}\right)$, Re $(\beta') > \left(\frac{m-1}{2}\right)$, $\| X \| < 1$, $\| Y \| < 1$.

4. APPELL FUNCTION F_2 OF MATRIX ARGUMENTS

The Appell function F_2 of matrix arguments represented by $F_2(\alpha, \beta, \beta'; \gamma, \gamma'; X, Y)$ comparable to the corresponding scalar case and defined by the integral.

$$ F_2(\alpha, \beta, \beta'; \gamma, \gamma'; X, Y) = \frac{\Gamma_m(\gamma)\,\Gamma_m(\gamma')}{\Gamma_m(\beta)\,\Gamma_m(\beta')\,\Gamma_m(\gamma-\beta)\,\Gamma_m(\gamma'-\beta')} $$

$$ \times \int_0^I\!\!\int_0^I |U|^{\beta-(m+1)/2} |V|^{\beta'-(m+1)/2} |I - U|^{\gamma-\beta-(m+1)/2} $$

$$ \times |I - V|^{\gamma'-\beta'-(m+1)/2} |I - X^{1/2} U X^{1/2} - Y^{1/2} VY^{1/2}|^{-\alpha} dU\, dV \qquad ...(4.1) $$

where Re $(\beta, \beta', \gamma - \beta, \gamma' - \beta') > \frac{m-1}{2}$, $U = U'$, $V = V'$, $X = X' > 0$, $Y = Y' > 0$, $\| X \| + \| Y \| < 1$.

It should be noted that if we substitute $H = X^{1/2} L^{-1}$, $K = Y^{1/2} M^{-1}$ then $H'H = 1$, $K'K = 1$, so we say that H and K are orthogonal. Substituting $X^{1/2} U X^{1/2} = L'H' U HL$, $Y^{1/2} V Y^{1/2} = M'K' V KM$ in (4.1), and transforming to $R = H' U H$, $S = K' V K$, we obtain the definition (4.1) in terms of

$$| I - L' UL - M' VM |^{-\alpha} \qquad \ldots(4.2)$$

where L, M are any matrices such that L'L = X, M'M = Y.

Theorem 3 –

$$F_2(\alpha, \beta, \beta'; \gamma, \gamma'; X, Y) = \frac{1}{\Gamma_m(\alpha)} \int_0^I |R|^{\alpha-(m+1)/2} \exp(-\mathrm{tr}R)$$

$$\times \ {}_1F_1(\beta;\gamma;\ X^{1/2}\,R\,X^{1/2})\ {}_1F_1(\beta';\gamma';\ Y^{1/2}\,R\,Y^{1/2})\,dR. \qquad \ldots(4.3)$$

where, Re $(\alpha) > (m-1)/2$, $X = X' > 0$, $Y = Y' > 0$, $\| X \| + \| Y \| < 1$.

Proof: If we substitute for $| I - X^{1/2} U X^{1/2} - Y^{1/2} V Y^{1/2}|^{-\alpha}$ on the right-hand side of (4.1) and we interchange the order of integration, which will be possible due to the condition of the theorem and later we integrate U and V in view of result

$${}_1F_1(\alpha;\beta;X^{1/2}\ Z\,X^{1/2}) = \frac{\Gamma_m(\beta)}{\Gamma_m(\alpha)\Gamma_m(\beta-\alpha)} \int_0^I |U|^{\alpha-(m+1)/2}|I-U|^{\beta-\alpha-(m+1)/2} \exp\left(trX^{\frac{1}{2}}ZX^{\frac{1}{2}}U\right) dU \qquad ..(4.4)$$

where Re $(\beta) > \left(\frac{m-1}{2}\right)$, Re $(\alpha) > \left(\frac{m-1}{2}\right)$, Re $(\beta-\alpha) > \left(\frac{m-1}{2}\right)$ and $\|XZ\| < 1$;

Also, $\mathrm{Tr}(X^{1/2} Z X^{1/2} U) = \mathrm{Tr}\,(U^{1/2} X^{1/2} Z X^{1/2} U^{1/2})$ and then we obtain (4.3).

Theorem 4 –

$$F_2(\alpha, \beta, \beta'; \gamma, \gamma'; X, Y)$$

$$= \frac{\Gamma_m(\delta)}{\Gamma_m(\alpha)\Gamma_m(\delta-\alpha)} \int_0^I |Z|^{\alpha-(m+1)/2}\,| I - Z |^{\delta-\alpha-(m+1)/2}\, F_2(\delta, \beta, \beta'; \gamma, \gamma'; Z^{1/2} XZ^{1/2}, Z^{1/2} YZ^{1/2})\,dZ. \qquad \ldots(4.5)$$

where, $R(\alpha) > (m-1)/2$, $R(\delta) > (m-1)/2$, $R(\delta-\alpha) > (m-1)/2$ and $\| X \| + \| Y \| < 1$.

Proof: Using double integration definition for F_2 and applying (4.1), we have

$$F_2(\delta, \beta, \beta'; \gamma, \gamma'; Z^{1/2} X Z^{1/2}, Z^{1/2} Y Z^{1/2}) = \frac{\Gamma_m(\gamma)\,\Gamma_m(\gamma')}{\Gamma_m(\beta)\,\Gamma_m(\beta')\,\Gamma_m(\gamma-\beta)\,\Gamma_m(\gamma'-\beta')}$$

$$\times \int_0^I\int_0^I | U |^{\beta-(m+1)/2} | V |^{\beta'-(m+1)/2}$$

$$\times |I-U|^{\gamma-\beta-(m+1)/2}\,|I - V|^{\gamma'-(m+1)/2}| I - Z^{1/2} X^{1/2} U X^{1/2} Z^{1/2} - Z^{1/2} Y^{1/2} V Y^{1/2} Z^{1/2})\,dU\,dV. \qquad \ldots(4.6)$$

Applying (4.6) on the R.H.S of (4.5) and integrate Z by using the result (3.3),we will obtain our result.

Theorem 5 –

$$F_2(\alpha, \beta, \beta'; \gamma, \gamma'; X, Y) = \frac{\Gamma_m(\delta)\,\Gamma_m(\delta')}{\Gamma_m(\beta)\,\Gamma_m(\beta')\,\Gamma_m(\delta-\beta)\,\Gamma_m(\delta'-\beta')}$$

$$\times \int_0^I \int_0^I |U|^{\beta-(m+1)/2} |V|^{\beta'-(m+1)/2}$$

$$\times |I-U|^{\delta-\beta-(m+1)/2} |I-V|^{\delta'-\beta'-(m+1)/2} \times F_2(\alpha, \delta, \delta'; \gamma, \gamma'; X^{1/2} U X^{1/2}, Y^{1/2} V Y^{1/2})\, dU\, dV. \quad ...(4.7)$$

where Re (δ), Re (δ'), Re (β), Re (β'), Re (γ), Re (γ') > (m – 1)/2, Re (δ – β), Re (δ' – β'), Re (γ – δ),

Re (γ' – δ') > (m – 1)/2 , $\| X \| < 1$ and $\| Y \| < 1$,

Proof: This result can be proved easily by using theorem 3 and the following result

$$\int_0^I U^{\beta-(m+1)/2} |I-U|^{\delta-\beta-(m+1)/2} \; {}_1F_1\left[\delta;\ \gamma;\ R^{1/2} X^{1/2} U X^{1/2} R^{1/2}\right] dU$$

$$= \frac{\Gamma_m(\beta)\,\Gamma_m(\delta-\beta)}{\Gamma_m(\delta)} \, {}_1F_1\left[\beta; \delta; R^{1/2} X R^{1/2}\right] \quad ...(4.8)$$

Theorem 6 –

$$F_2(\alpha, \beta, \beta'; \gamma, \gamma'; X, Y) = \frac{\Gamma_m(\gamma)\,\Gamma_m(\gamma')}{\Gamma_m(\delta)\,\Gamma_m(\delta')\,\Gamma_m(\gamma-\delta)\,\Gamma_m(\gamma'-\delta')}$$

$$\times \int_0^I \int_0^I |U|^{\delta-(m+1)/2} |V|^{\delta'-(m+1)/2}$$

$$\times |I-U|^{\gamma-\delta-(m+1)/2} |I-V|^{\gamma'-\delta'-(m+1)/2} F_2(\alpha, \beta, \beta'; \delta, \delta'; X^{1/2} U X^{1/2}, Y^{1/2} V Y^{1/2})\, dU\, dV. \quad ...(4.9)$$

Proof: On the right-hand side of (4.9), if we replace F_2 using Theorem 5, it will give us

$$\Sigma_1 = X_1 \int_0^I\int_0^I\int_0^I\int_0^I |U|^{\beta-(m+1)/2} |V|^{\beta'-(m+1)/2} |I-U|^{\delta-\beta-(m+1)/2}$$

$$\times |I-V|^{\delta'-\beta'-(m+1)/2} |R|^{\delta-(m+1)/2} |S|^{\delta'-(m+1)/2} |I-R|^{\gamma-\delta-(m+1)/2}$$

$$\times |I-S|^{\gamma'-\delta'-(m+1)/2} |1 - X^{1/2} R^{1/2} U R^{1/2} X^{1/2}$$

$$- X^{1/2} S^{1/2} V S^{1/2} Y^{1/2}|^{-\alpha}\, dU\, dV\, dR\, dS \quad ...(4.10)$$

where $X_1 = \dfrac{\Gamma_m(\gamma)\,\Gamma_m(\gamma')}{\Gamma_m(\beta)\,\Gamma_m(\beta')\,\Gamma_m(\delta'-\beta')\,\Gamma_m(\delta-\beta)\Gamma_m(\gamma-\delta)\Gamma_m(\gamma'-\delta')}$

Consider the transformation of U to Λ_1 and V to Λ_2 through

$R^{1/2} U R^{1/2} = \Lambda_1 \Rightarrow \quad |R|^{(m+1)/2} dU = d\Lambda_1$

$$S^{1/2}\, V\, S^{1/2} = \Lambda_2 \;\Rightarrow\; \quad |\, S\, |^{(m+1)/2}\, dV = d\Lambda_2$$

$$0 < \Lambda_1 < R,\; 0 < \Lambda_2 < S \text{ and}$$

$$U = R^{-1/2}\, \Lambda_1\, R^{-1/2},\; V = S^{-1/2}\, \Lambda_2\, S^{-1/2}.$$

Substituting the above transformation in (4.10), we observe that

$$\Sigma_1 = \quad X_1 \int_{\Lambda_1}\int_{\Lambda_2}\int_{R}\int_{S} |\, \Lambda_1\, |^{\beta - (m+1)/2}\, |\, \Lambda_2\, |^{\beta' - (m+1)/2}\; |\, R - \Lambda_1\, |^{\delta - \beta - (m+1)/2}\, |\, S - \Lambda_2\, |^{\delta' - \beta' - (m+1)/2}$$

$$x\, |\, I - R\, |^{\gamma - \delta - (m+1)/2}\, |\, I - S\, |^{\gamma' - \delta' - (m+1)/2}\, |\, I - X^{1/2}\, \Lambda_1\, X^{1/2} - Y^{1/2}\, \Lambda_2\, Y^{1/2}\, |^{-\alpha}\, d\, \Lambda_1\, d\, \Lambda_2\, dR\, dS. \quad ...(4.11)$$

The integration is taken over Λ_1, Λ_2, R and S such that $0 < \Lambda_1 < R$, $0 < \Lambda_2 < S$, $0 < R << I$ and $0 < S < I$ respectively.

Here $\quad |\, R - \Lambda_1\, | = |\, (I - \Lambda_1) - (I - R)\, |,\; |\, S - \Lambda_2\, | = |\, (I - \Lambda_2) - (I - S)\, |$

Put $\quad I - R = (I - \Lambda_1)^{1/2}\, Z_1\, (I - \Lambda_1)^{1/2}, \quad dR = |\, I - \Lambda_1\, |^{(m+1)/2}\, dZ_1$

$$I - S = (I - \Lambda_2)^{1/2}\, Z_2\, (I - \Lambda_2)^{1/2}, \quad dR = |\, I - \Lambda_2\, |^{(m+1)/2}\, dZ_2$$

The above substitutions in (4.10) yield

$$\Sigma_1 = \quad X_1 \int_0^I\int_0^I |\, \Lambda_1\, |^{\beta - (m+1)/2}\, |\, \Lambda_2\, |^{\beta' - (m+1)/2}\, |\, I - \Lambda_1\, |^{\gamma - \beta - (m+1)/2}$$

$$x\, |\, I - \Lambda_2\, |^{\gamma' - \beta' - (m+1)/2}\, |\, I - X^{1/2}\, \Lambda_1\, X^{1/2} - Y^{1/2}\, \Lambda_2\, Y^{1/2}\, |^{-\alpha}\, d\, \Lambda_1\, d\, \Lambda_2$$

$$x \int_0^I\int_0^I |\, Z_1\, |^{\gamma - \beta - (m+1)/2}\, |\, Z_2\, |^{\gamma' - \beta' - (m+1)/2}\, |\, I - Z_1\, |^{\delta - \beta - (m+1)/2}$$

$$x\, |\, I - Z_2\, |^{\gamma' - \beta' - (m+1)/2}\, dZ_1\, dZ_2. \quad ...(4.12)$$

Now analysing (4.10) in view of the result Herz[4]

$$B_m\, (\alpha, \beta) = \int_0^I |\, X\, |^{\alpha - (m+1)/2}\, |\, I - X\, |^{\beta - (m+1)/2}\, dX = \frac{\Gamma_m(\alpha)\Gamma_m(\beta)}{\Gamma_m(\alpha+\beta)} \quad ...(4.13)$$

$$\text{provided,} \qquad \mathrm{Re}\,(\alpha),\, \mathrm{Re}\,(\beta) > \left(\frac{m-1}{2}\right)$$

The theorem can be proved easily.

Alternative Proof: As pointed out by the referee, a simpler proof of Theorem 6 can be worked out as follows. Note that in Theorem 3, since ${}_1F_1$ (like ${}_1F_0$, ${}_2F_1$ etc.) is a function of the latest roots of the argument, we may also take $R^{1/2} X R^{1/2}$, $R^{1/2} Y R^{1/2}$ instead of $X^{1/2} R X^{1/2}$ and $Y^{1/2} R Y^{1/2}$. Hence the result (4.9) follows simply from the integral

$$\int_0^I | U |^{\delta - (m+1)/2} | I - U |^{\gamma - \delta - (m+1)/2} \; {}_1F_1 (\beta; \delta; R^{1/2} X^{1/2} U X^{1/2} R^{1/2}) \, dU$$

$$= \frac{\Gamma_m(\delta)\,\Gamma_m(\gamma-\delta)}{\Gamma_m(\gamma)} \; {}_1F_1 (\beta; \gamma; R^{1/2} X^{1/2}), \qquad \text{...(4.14)}$$

where $\mathrm{Re}\,(\delta) > \left(\frac{m-1}{2}\right)$, $\mathrm{Re}\,(\gamma - \delta) > \left(\frac{m-1}{2}\right)$.

6. CONCLUDING REMARK

As indicated by the referee, it is interesting to observe that it follows easily from the definitions that the Appell functions discussed in this paper are invariant under the simultaneous transformations

$$X \to H' \; XH, \quad Y \to H' \, YH \qquad \text{... (6 .1)}$$

by an orthogonal matrix H. Hence their series expansions will involve not only terms of the form $tr(X^r), tr(Y^s)$ which are expressible in terms of latent roots of X and Y, but also such terms as $tr(XY)$ $tr(X^2Y)$, $tr(X^2 \, Y^2)$. $tr(XYXY)$ etc. which are also invariant under (6.1) but are not expressible in terms of roots of X and Y. However, the form of these latter terms shows that Appell function arguments of the form $(U^{1/2} XU^{1/2}, U^{1/2} YU^{1/2})$ may be written without ambiguity as (XU, YU) or (UX, UY) but not as $(X^{1/2}, UX^{1/2}, Y^{1/2} UY^{1/2})$, (XU, UY), or (UX, YU). But it must be emphasized that for Appell functions, as well as for the ${}_pF_q$, the unsymmetric arguments should only be used for concise statement of results in calculations, the symmetric forms should be strictly used.

Expansions of analytic functions with the invariance property (6.1) may be possible in terms of invariant polynomials introduced as done in Wang[7] and Ozarslan[9]. The invariant polynomials are generalizations of the Zonal polynomials to two matrix arguments with the invariance property (6.1). It is believed that the series expansions of the Appell functions may be expressed in terms of these polynomials. Since the series expansions of Appell functions would give some completeness to our work hence it will form the· subject matter of a future communication.

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