



# Assessing the Efficacy of FRDTM and FDM in Solving Time Fractional Telegraph Equations: A Comparative Study

<sup>1</sup>Nitin Sachan,<sup>2</sup>Diwari Lal

<sup>1</sup>Research Scholar,<sup>2</sup>Professor

<sup>1-2</sup>Narain (PG) College, Shikohabad, Uttar Pradesh (India)

Affiliated to Dr. Bhimrao Ambedkar University, Agra, Uttar Pradesh (India)

**Abstract:** This paper gives a comparative evaluation of the effectiveness of two numerical methods, namely the Fractional Reduced Differential Transform Method (FRDTM) and the Finite Difference Method (FDM), in the process of solving the time fractional telegraph equation (TFTE). The TFTE is a mathematical model that has numerous applications in a variety of domains, including engineering and physics, among others. The purpose of this study is to investigate and evaluate the performance of FRDTM and FDM with regard to the accuracy, stability, and computational efficiency of the two methods. First, a concise introduction to the TFTE and the significance of its solutions in practical applications is provided. This is followed by the main body of the paper. Following that, the numerical techniques known as FRDTM and FDM are presented and described in further detail. A series of numerical experiments are carried out in order to carry out the comparison analysis. These experiments take into consideration a variety of scenarios and parameter values. To decide the degree of precision, the mathematical arrangements are analyzed against insightful arrangements that are now known. The results of the comparison study provide useful insights into the strengths and limits of FRDTM and FDM when applied to time fractional telegraph equations. Those academics and practitioners who are looking for optimal numerical solutions for issues that are governed by such equations will find this information to be critically important. The research makes a contribution to the continuing discussion over the selection of acceptable numerical methods for fractional differential equations. It also advances our understanding of the application and performance of these approaches in situations that occur in the real world.

**Keywords:** TFTE, FRDTM, Approximation Techniques, Efficient Approximations, Fractional Derivative

## I. INTRODUCTION

Knowledge the dynamic behavior of physical systems that exhibit anomalous diffusion and memory effects requires a study of time fractional telegraph equations, which plays a significant part in this knowledge. Because of the many applications that may be found in the fields of science and engineering, the numerical solution of these equations needs to be accurate and efficient. FRDTM and the FDM are two separate mathematical techniques that are the subject of this examination. The motivation behind this exploration is to direct a near report with the target of assessing the viability of these two quantitative methodologies. FRDTM is a recently developed numerical technique that has demonstrated potential in the management of fractional differential equations. The FDM is a well-established numerical approach that has been extensively utilized for solving partial differential equations. The fundamental objective of this comparative study is to give a full evaluation and comparison of the performance of FRDTM and FDM throughout the process of solving TFTE. This is the primary purpose of this comparative study. The accuracy, stability, and computational efficiency of each approach will be thoroughly investigated as part of this process. The effectiveness of these methods to capture the detailed dynamics of time fractional telegraph equations will be evaluated through the use of numerical simulations. These simulations will be carried out under a variety of scenarios by adjusting the parameters.

Utilizing brain network procedures, Lagaris et al. (1998) had the option to settle conventional differential conditions as well as fractional differential conditions. To fulfill the limit necessities, the preliminary arrangements that have been given have been fixed.

To address fragmentary differential conditions, Pang et al. (2018) supported the utilization of brain networks that were directed by partial physical science. Settling fragmentary request broadcast conditions was achieved by Khan et al. (2019) using the Laplace-Adomian decomposition technique. They made use of the Caputo operator to define the fractional derivative. Utilizing the proposed technique, they had the option to infer series structure answers for fragmentary request transmit conditions. To address the Summed up Time-Fragmentary Message Condition (GTFTE), Kumar et al. (2019) suggested a method based on finite differences. Using GFD terms, the equation was defined using this method. To lay out a semi-logical answer for fragmentary incomplete differential conditions, the Generalized Differential Transformational method, also known as the GDTM, was proposed by Mohammadian et al. (2019). Ramani et al. (2019) found solutions to the fractional wave equations in three dimensions and the time fractional heat equations in two dimensions. They also created and executed a summed up FRDTM. Using graphical techniques, for example, the stage plot of the surmised answer for II and the mistake chart, they exhibited the mathematical productivity of the interaction that was proposed. In their work from 2020, Wu and Yang fostered the exchanging section unadulterated express understood and verifiable unequivocal equal contrast strategies for the time fragmentary message condition. Using hypothetical investigation, they arrived at the resolution that the plans are both genuinely steady and concurrent. In order to address the Boltzmann transport condition that is reliant on the passage of time, Hamada (2020) came up with an innovative model to suggest. An adjusted disintegration strategy known as the Sumudu-Adomian Deterioration Technique (SADM) was used by Devi and Jakhar (2021) to tackle partial request broadcast conditions. To tackle the time-partial message condition, Ibrahim and Bijiga (2021) used a brain network approach. They constructed a misfortune capability that included starting and limit conditions, as well as boundaries (loads and predispositions) that could be freely changed. Utilizing the straight space-partial message condition, Alaofi et al. (2020) gave a conventional structure that can be utilized to direct quadratic spline capabilities to build a mathematical methodology for tracking down a guess arrangement. The numerical method can be developed with the help of this framework as a guide. What's more, they used the Von Neumann approach to assess the solidness of the logical plan, which exhibited that the technique that was proposed is reasonable for use under specific circumstances.

Through the utilization of a natural transform in one, two, and three dimensions, Mamta and Samanyu (2023) were able to solve time-fractional hyperbolic telegraph equations in an iterative manner. The feeling of fragmentary subordination that Caputo had felt was something that they considered. The solution to the problem of the time-partial message condition was discovered by Sahraee and Arabameri (2023) through the application of a methodology known as the half and half method. For the purpose of the mathematical reenactment, a combination of the limited contrast and differential change techniques was successfully used. The condition was initially semi-discretized along the spatial ordinate and was initially described. As a result, the strategy of partial differential change was utilized in order to handle the arrangement of conventional differential conditions, which began with the semi-discretization of the scenario. This research is significant because it offers useful insights into the strengths and limitations of FRDTM and FDM for solving TFTE. This discovery is the reason for the significance of this research. It is possible for academics and practitioners to make educated decisions regarding the selection of a numerical approach based on the specific requirements of their applications provided they have a thorough understanding of the comparative advantages of each numerical approach. It is anticipated that the findings of this research will make a contribution to the continuous development and refining of numerical approaches for solving fractional differential equations, which will ultimately improve our capacity to accurately describe and simulate complicated physical systems.

## II. FINITE DIFFERENCE TECHNIQUES FOR SOLVING TFTE

There is a fragmentary subsidiary in time that is remembered for the time fragmentary message condition, which is a fractional differential condition connected with time. A mathematical strategy that is regularly used to estimated answers for partial differential conditions is known as the finite difference method (FDM). We make benefit of the Grunwald-Letnikov meaning of the fragmentary subsidiary to discretize the time partial message condition by utilizing the FDM.

Coming up next is an articulation that can be utilized to communicate the one-layered time fragmentary message condition with introductory condition:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 2\alpha \frac{\partial u}{\partial x} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad (1)$$

$u(x, t)$  is the reliant variable,  $x$  is the spatial variable, and  $t$  is the time variable. The partial request is composed as  $0 \leq \alpha \leq 1$ , and  $c$  is a steady.

Starting now and into the foreseeable future, we will utilize the limited contrast strategy to discretize both the reality spaces. Allow us to expect that the time step size is indicated by  $\Delta t$ , and the spatial step size is meant by  $\Delta x$ . To communicate the discretized conditions for the time partial message condition, the accompanying recipe can be utilized:

Step 1: Through the use of the Grunwald-Letnikov definition, discretize the time subsidiary.

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \sum_{n=0}^k (-1)^k \binom{\alpha}{k} u(x, t - k\Delta t) \quad (2)$$

Step 2: The spatial derivatives should be discretized by applying the central differences equation:

$$\frac{\partial u}{\partial x} = \frac{u(x+\Delta x, t) - u(x-\Delta x, t)}{2\Delta x} \quad (3)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x+\Delta x, t) - 2u(x, t) + u(x-\Delta x, t)}{(\Delta x)^2} \quad (4)$$

Make these discretizations a part of the original FTE (1), and then solve the resulting system of equations numerically by making use of the suitable initial condition.

There is a possibility that the choice of discretization parameters and numerical methods will have an effect on the stability and correctness of the quantitative solution. Adjusting these parameters and making use of boundary conditions that are adequate is absolutely necessary in order to achieve numerical solutions that are dependable.

### III. THE FRACTIONAL REDUCED DIFFERENTIAL TRANSFORM METHOD: AN OVERVIEW OF ORIGINAL METHOD FOR RESOLVING PDE INVOLVING FRACTIONS

Within this section, we shall present the concept of the FRDTM in a manner that is accompanied with the associated content. For the purpose of this discussion, let us consider a capability  $u(x, t)$  that is both logical and persistently differentiable  $k$  times with respect to time  $t$  and space  $x$  in the area of interest that is being considered. Because of this, the capability  $u(x, t)$  has altered, resulting in a lower capability.

$$U_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[ \frac{\partial t^{\alpha k}}{\partial t^{\alpha k}} \right]_{t=t_0} \quad (5)$$

Using the formula, we can determine the differential inverse transform of  $U_k(x)$ .

$$u(x, t) = \sum_{k=0}^{\infty} U_k(x) t^{\alpha k} \quad (6)$$

Table 1: Fundamental functions of the FRDTM

Functional form	Reduced transformed function
$\frac{\partial^{n\alpha} u}{\partial t^{n\alpha}}$	$\frac{\Gamma(k\alpha+n\alpha+1)}{\Gamma(k\alpha+1)} U_{k+n}(x)$
$\frac{\partial^n u}{\partial x^n}$	$\frac{\partial^n}{\partial x^n} U_k(x)$
$x^m t^n u(x, t)$	$x^m U_{k-n}(x)$
$x^m t^n$	$x^m \delta(k\alpha - n)$

#### IV. UTILIZING THE FRDTM TO ADDRESS THE TFTE

TFTE can be solved numerically using the Fractional Derivative (FD) method, which is often referred to as the Fractional Derivative Time-stepping Method (FRDTM). The propagation of signals in a medium having fractional derivatives in the time domain is described by the partial differential equation known as the time fractional telegraph equation." Because it can record fractional order derivatives with accuracy-a feature that is crucial for simulating anomalous diffusion or non-local transport phenomena-the FRDTM is especially good at solving these kinds of equations. Time fractional telegraph equations can be effectively solved computationally with this approach, which discretizes and approximates the fractional derivatives using the concepts of fractional calculus. It is a solid choice for demonstrating and grasping the complicated elements of frameworks with partial request differential conditions, such the time fragmentary message condition. It does this by safeguarding mathematical solidness and accuracy.

**Numerical illustration 4.1:** Using the Caputo fractional derivative, let us examine the TFTE. The following is one way to write the equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \beta \frac{\partial^\gamma u}{\partial t^\gamma} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad (7)$$

$$u(x, 0) = e^{-x^2} \quad (8)$$

where  $\frac{\partial^\alpha u}{\partial t^\alpha}$  is the Caputo fractional derivative with respect to time,  $\frac{\partial^2 u}{\partial x^2}$  is the second-order spatial derivative,  $\alpha$  and  $\gamma$  are time fractional orders and  $\beta$  is constant.

When we apply differential transform on both sides, we obtain

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)} U_{k+1}(x) + \beta \frac{\Gamma(k\gamma+\gamma+1)}{\Gamma(k\gamma+1)} U_{k+1}(x) = c^2 \frac{\partial^2}{\partial x^2} U_k(x)$$

$$U_{k+1}(x) \left[ \frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)} + \beta \frac{\Gamma(k\gamma+\gamma+1)}{\Gamma(k\gamma+1)} \right] = c^2 \frac{\partial^2}{\partial x^2} U_k(x)$$

$$U_{k+1}(x) \left[ \frac{\Gamma(k\alpha+\alpha+1)\Gamma(k\gamma+1) + \beta\Gamma(k\gamma+\gamma+1)\Gamma(k\alpha+1)}{\Gamma(k\alpha+1)\Gamma(k\gamma+1)} \right] = c^2 \frac{\partial^2}{\partial x^2} U_k(x)$$

$$U_{k+1}(x) = c^2 \frac{\Gamma(k\alpha+1)\Gamma(k\gamma+1)}{\Gamma(k\alpha+\alpha+1)\Gamma(k\gamma+1) + \beta\Gamma(k\gamma+\gamma+1)\Gamma(k\alpha+1)} \frac{\partial^2}{\partial x^2} U_k(x)$$

$$k = 0, U_1(x) = c^2 \frac{1}{\Gamma(\alpha+1) + \beta\Gamma(\gamma+1)} \frac{\partial^2}{\partial x^2} U_0(x) = \frac{c^2}{\Gamma(\alpha+1) + \beta\Gamma(\gamma+1)} \frac{\partial^2}{\partial x^2} e^{-x^2}$$

$$= \frac{c^2}{\Gamma(\alpha+1) + \beta\Gamma(\gamma+1)} (4x^2 - 2) e^{-x^2}$$

$$k = 1, U_2(x) = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)c^2}{\Gamma(2\alpha+1)\Gamma(\gamma+1) + \beta\Gamma(2\gamma+1)\Gamma(\alpha+1)} \frac{\partial^2}{\partial x^2} U_1(x)$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)c^2}{\Gamma(2\alpha+1)\Gamma(\gamma+1) + \beta\Gamma(2\gamma+1)\Gamma(\alpha+1)} \frac{\partial^2}{\partial x^2} \frac{c^2}{\Gamma(\alpha+1) + \beta\Gamma(\gamma+1)} (4x^2 - 2) e^{-x^2}$$

$$= \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)c^4}{[\Gamma(2\alpha+1)\Gamma(\gamma+1) + \beta\Gamma(2\gamma+1)\Gamma(\alpha+1)][\Gamma(\alpha+1) + \beta\Gamma(\gamma+1)]} (16x^4 - 48x^2 + 12) e^{-x^2}$$

Because of this, the answer to the equation for the time fractional telegraph is

$$u(x, t) = \sum_{k=0}^{\infty} U_k(x) t^{\alpha k} = U_0(x) + U_1(x) t^\alpha + U_2(x) t^{2\alpha} + \dots$$

$$u(x, t) = e^{-x^2} + \frac{c^2}{\Gamma(\alpha+1) + \beta\Gamma(\gamma+1)} (4x^2 - 2) e^{-x^2} t^\alpha + \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)c^4}{[\Gamma(2\alpha+1)\Gamma(\gamma+1) + \beta\Gamma(2\gamma+1)\Gamma(\alpha+1)][\Gamma(\alpha+1) + \beta\Gamma(\gamma+1)]} (16x^4 - 48x^2 + 12) e^{-x^2} t^{2\alpha} +$$

...

(9)

**Numerical illustration 4.2:** A partial differential equation that incorporates a fractional derivative in time is referred to as the TFTE. That condition can be communicated as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 2\alpha \frac{\partial u}{\partial x} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad (10)$$

$$u(x, 0) = \sin \pi x \quad (11)$$

The fractional order is denoted by  $0 \leq \alpha \leq 1$ , while the wave speed is denoted by  $c$ .

$$\frac{\partial^{n\alpha} u}{\partial t^{n\alpha}} = \frac{\Gamma(k\alpha + n\alpha + 1)}{\Gamma(k\alpha + 1)} U_{k+n}(x)$$

$$\frac{\partial^n u}{\partial x^n} = \frac{\partial^n}{\partial x^n} U_k(x)$$

Taking Differential transform on both sides, we get

$$\frac{\Gamma(k\alpha + \alpha + 1)}{\Gamma(k\alpha + 1)} U_{k+1}(x) + 2\alpha \frac{\partial}{\partial x} U_k(x) = c^2 \frac{\partial^2}{\partial x^2} U_k(x)$$

$$\frac{\Gamma(k\alpha + \alpha + 1)}{\Gamma(k\alpha + 1)} U_{k+1}(x) = c^2 \frac{\partial^2}{\partial x^2} U_k(x) - 2\alpha \frac{\partial}{\partial x} U_k(x)$$

$$\frac{\Gamma(k\alpha + \alpha + 1)}{\Gamma(k\alpha + 1)} U_{k+1}(x) = \left[ c^2 \frac{\partial^2}{\partial x^2} - 2\alpha \frac{\partial}{\partial x} \right] U_k(x)$$

$$U_{k+1}(x) = \frac{\Gamma(k\alpha + 1)}{\Gamma(k\alpha + \alpha + 1)} \left[ c^2 \frac{\partial^2}{\partial x^2} - 2\alpha \frac{\partial}{\partial x} \right] U_k(x)$$

$$k = 0, U_1(x) = \frac{1}{\Gamma(\alpha + 1)} \left[ c^2 \frac{\partial^2}{\partial x^2} - 2\alpha \frac{\partial}{\partial x} \right] U_0(x) = \frac{1}{\Gamma(\alpha + 1)} \left[ c^2 \frac{\partial^2}{\partial x^2} - 2\alpha \frac{\partial}{\partial x} \right] \sin \pi x$$

$$U_1(x) = \frac{1}{\Gamma(\alpha + 1)} [-c^2 \pi^2 \sin \pi x - 2\alpha \pi \cos \pi x] = -\frac{1}{\Gamma(\alpha + 1)} [c^2 \pi^2 \sin \pi x + 2\alpha \pi \cos \pi x]$$

$$k = 1, U_2(x) = \frac{\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} \left[ c^2 \frac{\partial^2}{\partial x^2} - 2\alpha \frac{\partial}{\partial x} \right] U_1(x) = -\frac{\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} \left[ c^2 \frac{\partial^2}{\partial x^2} - 2\alpha \frac{\partial}{\partial x} \right] \frac{1}{\Gamma(\alpha + 1)} [c^2 \pi^2 \sin \pi x + 2\alpha \pi \cos \pi x]$$

$$= -\frac{1}{\Gamma(2\alpha + 1)} [-\pi^4 c^4 \sin \pi x - 2\alpha \pi^3 c^2 \cos \pi x + 4\alpha^2 \pi^2 \sin \pi x - 2\alpha \pi^3 c^2 \cos \pi x]$$

$$= \frac{1}{\Gamma(2\alpha + 1)} [\pi^4 c^4 \sin \pi x + 2\alpha \pi^3 c^2 \cos \pi x - 4\alpha^2 \pi^2 \sin \pi x + 2\alpha \pi^3 c^2 \cos \pi x]$$

$$= \frac{1}{\Gamma(2\alpha + 1)} [(\pi^4 c^4 - 4\alpha^2 \pi^2) \sin \pi x + 4\alpha \pi^3 c^2 \cos \pi x]$$

The result of applying the differential transform on both sides is

$$u(x, t) = \sum_{k=0}^{\infty} U_k(x) t^{\alpha k} = U_0(x) + U_1(x) t^{\alpha} + U_2(x) t^{2\alpha} + \dots$$

$$u(x, t) = \sin \pi x - \frac{1}{\Gamma(\alpha + 1)} [c^2 \pi^2 \sin \pi x + 2\alpha \pi \cos \pi x] t^{\alpha} + \frac{1}{\Gamma(2\alpha + 1)} [(\pi^4 c^4 - 4\alpha^2 \pi^2) \sin \pi x + 4\alpha \pi^3 c^2 \cos \pi x] t^{2\alpha} \quad (12)$$

## V. QUANTITATIVE REVELATIONS: RESULTS AND THEIR SIGNIFICANCE

Utilizing FRDTM, a three-dimensional solution to TFTE has been shown in the form of a numerical illustration (4.1) in graph (1). The graphs (2), (3), and (4) successively illustrate the variation in the solution of the numerical illustration (4.1) with spatial coordinate ( $x$ ) for various values of time coordinate ( $t$ ), fractional orders  $\alpha$ , and constant  $\beta$  achieved through the use of FRDTM.

Through the utilization of FRDTM, the numerical depiction (4.2) of the three-dimensional solution of the TFTE has been presented in graph (5).

The graphs (6) and (7), respectively, provide a visual representation of the variation in the solution of the numerical example (4.2) with spatial coordinate ( $x$ ) for different values of time coordinate ( $t$ ), fractional orders  $\alpha$ , and constant  $\beta$ . This variation is obtained by the utilization of FRDTM.

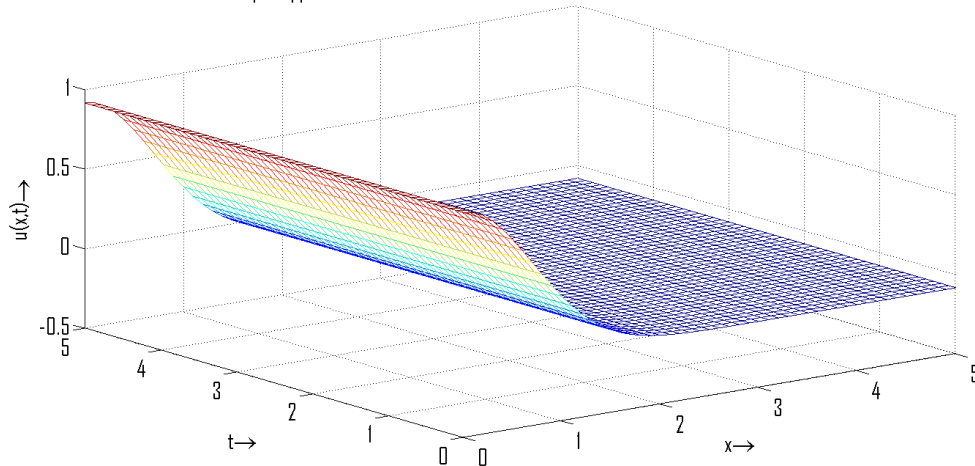
Understanding and improving numerical methods for fractional partial differential equations relies heavily on evaluating the effectiveness of FRDTM and FDM in solving Time Fractional Telegraph Equations (TFTE). The advantages and disadvantages of FRDTM and FDM can be better understood by comparing the two methods' outputs.

Findings from numerical analyses of FRDTM's TFTE simulations highlight the method's promise as a powerful computational tool for this type of challenge. However, a tried-and-true numerical approach that provides both computational efficiency and ease of use is the Finite Difference Method. The difficulties in properly expressing the non-local character of fractional derivatives have cast doubt on its usefulness in solving fractional differential equations. By comparing FDM with other

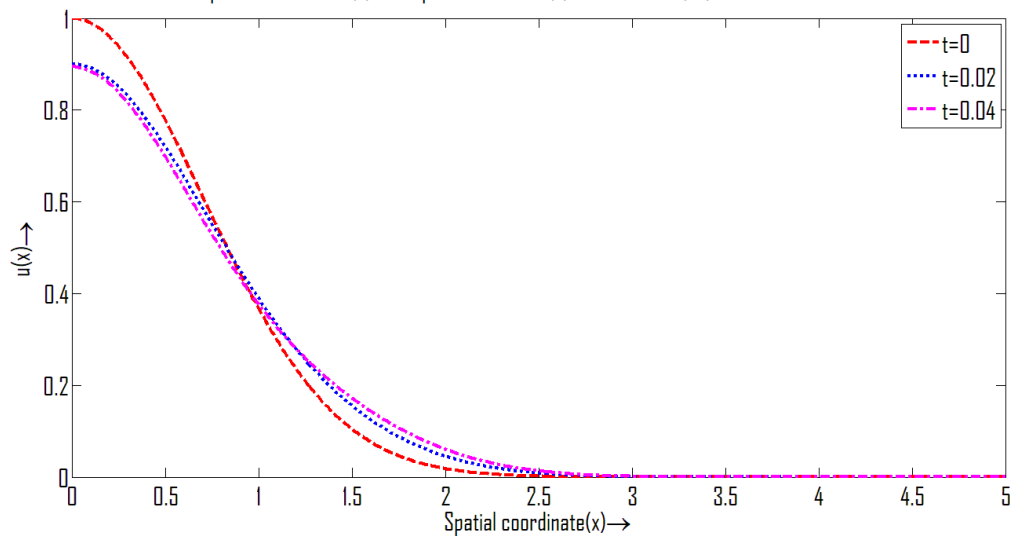
methods for solving TFTE, we may learn all about its strengths and weaknesses, which is helpful for researchers and practitioners who are looking for efficient numerical solutions to similar fractional differential equations.

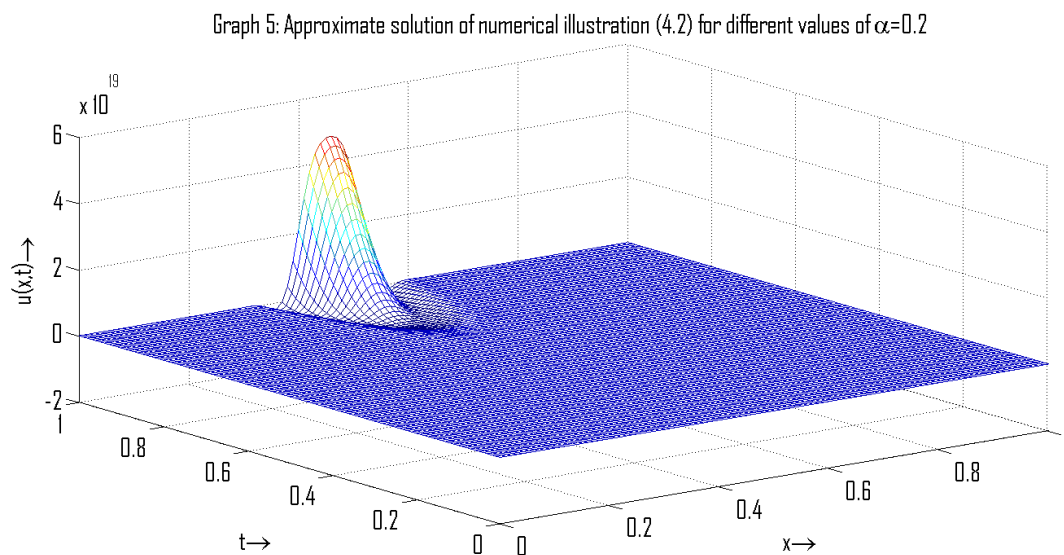
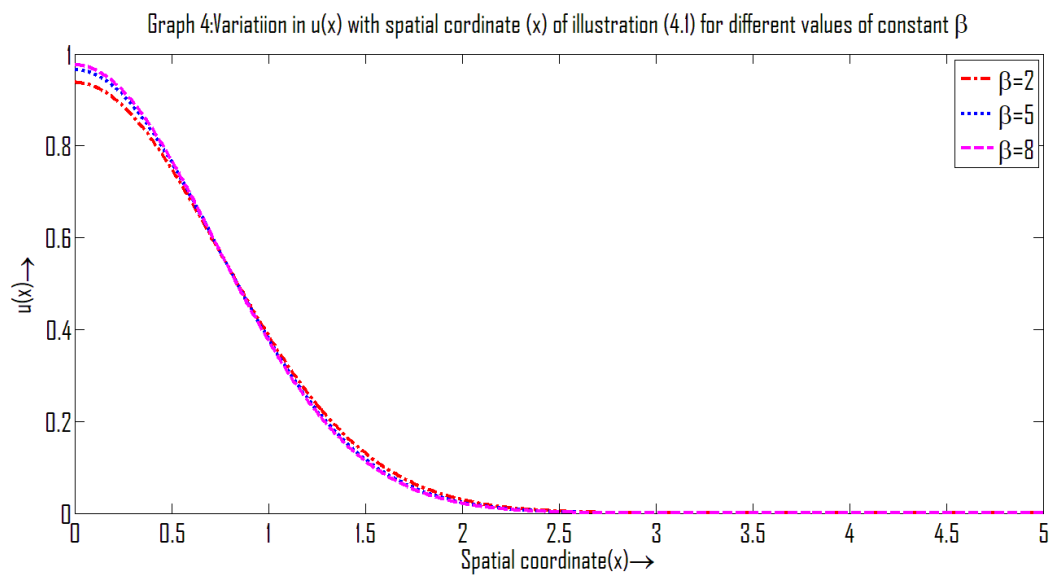
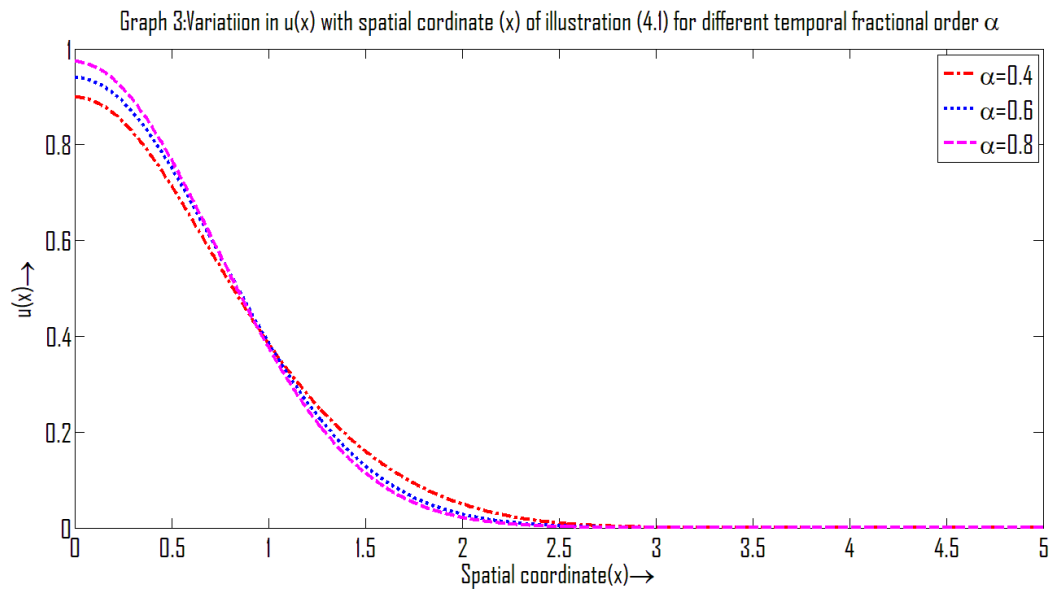
This near study is significant for mathematical examination of fragmentary halfway differential conditions by and large, not only for the TFTE issue specifically. The discoveries of this work can assist scientists and professionals with picking mathematical methodologies that are suitable for tackling partial differential conditions with specific properties. Eventually, the discoveries from contrasting FRDTM and FDM in settling TFTE add to the proceeding with tries to work on the exactness and productivity of mathematical strategies for fragmentary differential conditions, which could have ramifications for a scope of designing and logical fields.

Graph 1: Approximate solution of numerical illustration (4.1) for different values of  $\alpha=0.5$



Graph 2: Variation in  $u(x)$  with spatial coordinate ( $x$ ) of illustration (4.1) for different values of  $t$





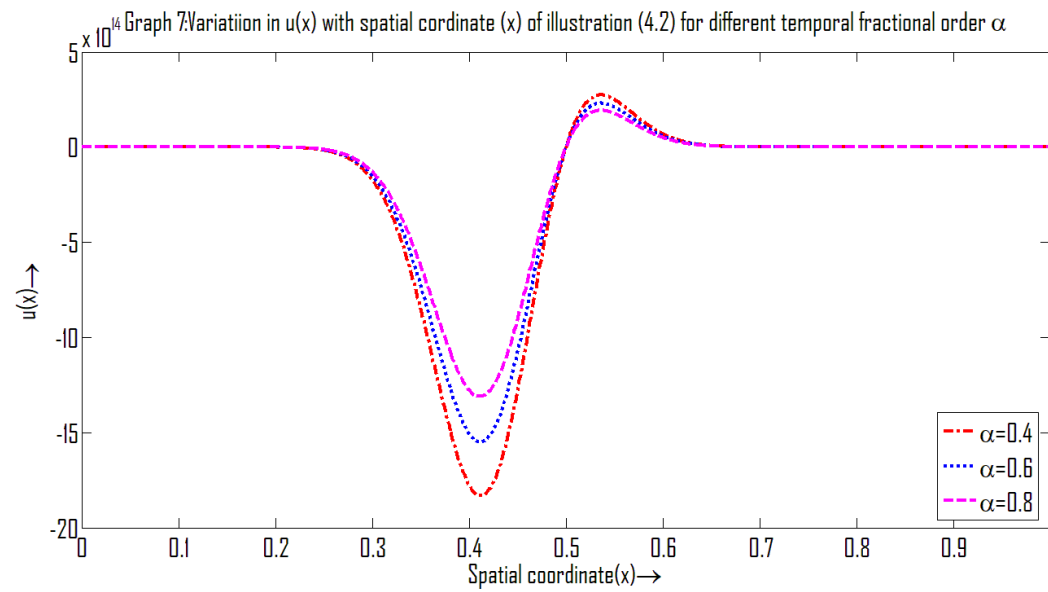
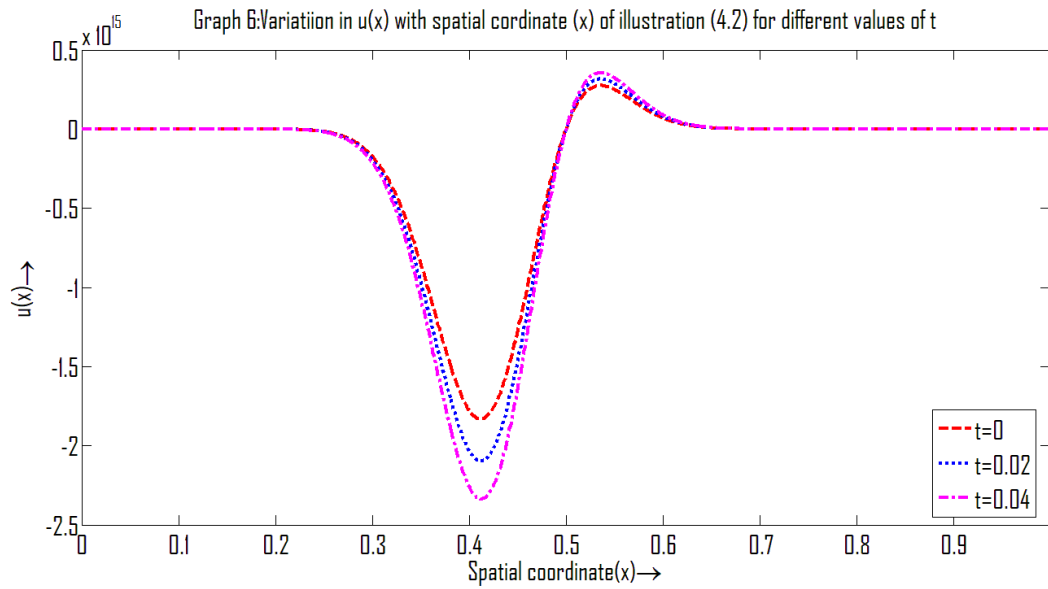


Table 2: Error Analysis between FRDTM and FDM solution of numerical illustration (4.1)

$x$	$t$	FRDTM	FDM	Error
0.1	0.2	0.9166	0.9154	0.0012
0.2	0.3	0.9411	0.9405	0.0006
0.3	0.4	0.8952	0.8937	0.0015
0.4	0.5	0.7592	0.7575	0.0017
0.5	0.6	0.5382	0.5371	0.0011
0.6	0.7	0.259	0.2585	0.0005
0.2	0.1	0.8289	0.8224	0.0065
0.3	0.2	0.8005	0.7881	0.0124
0.4	0.3	0.7205	0.7193	0.0012
0.5	0.4	0.5705	0.5683	0.0022
0.6	0.5	0.3617	0.3612	0.0005
0.7	0.6	0.1243	0.1221	0.0022

$x$	$t$	FRDTM	FDM	Error
0.4	0.6	0.0038	0.0021	0.0017
0.4	0.7	0.0063	0.0056	0.0007
0.5	0.6	0.0059	0.0047	0.0012
0.5	0.9	0.1291	0.1287	0.0004
0.6	0.9	0.0032	0.0026	0.0006
0.3	1	0.0043	0.0028	0.0015
0.4	1	2.0645	2.0633	0.0012
0.5	1	3.2107	3.2085	0.0022
0.6	1	0.0111	0.0102	0.0009

## VI. CONCLUDING REMARKS

Finally, the limits and strengths of each numerical approach have been illuminated by the comparative study on the usefulness of the FRDTM and FDM in solving time fractional telegraph equations. Under different scenarios, the numerical simulations have shown that FRDTM and FDM can accurately solve the time fractional telegraph equations. The comparison investigation has, nevertheless, revealed a few subtleties. Despite its relative youth, FRDTM has already proven to be an invaluable tool for solving fractional differential equations, because to its remarkable computing efficiency and flexibility in dealing with non-standard geometries. Its stochastic character provides a fresh angle on fractional derivatives, opening up new avenues for solving these problems. However, a well-tested numerical approach known as the Finite Difference Method has proven time and again that it can faithfully capture the dynamics of time fractional telegraph equations. It is a trusted option due to its familiarity and well-established structure, which is particularly useful in situations where convergence and stability are critical. The needs of the current challenge dictate whether FRDTM or FDM is better. It is possible that FRDTM might provide an advantage for applications that require very versatile and efficient computations. On the other hand, when a tried-and-true numerical method is desired, FDM is a reliable option, particularly when the problem's properties match the method's strengths. This research adds to our knowledge of numerical methods for fractional differential equations and gives academics and practitioners a more nuanced view on how to choose methods wisely. The results lay the groundwork for developing these numerical methods further, which will lead to better simulation and modeling capabilities for time fractional telegraph equations and other similar problems.

## VII. EXPLORATION OF POTENTIAL FUTURE RESEARCH AVENUES IN THIS PARTICULAR DIRECTION

Potentially fruitful directions for further research into the effectiveness of FRDTM and FDM in solving time fractional telegraph equations are numerous. Research into algorithmic refinement, which seeks to improve the precision and efficiency of both approaches, is still relevant. Optimizing computing resources could lead researchers to focus on developing adaptive mesh techniques for FDM and further refining the stochastic features of FRDTM. An interesting new trend is hybrid approaches, which aim to combine the best features of FRDTM and FDM to create a more flexible and effective strategy. Further, specific numerical methods may be developed as a result of insights into solution behavior in distinct domain regions that a multiscale study could offer. The study could be expanded to include higher dimensions and applied to real-world challenges to gain a better understanding of the methods' practicality and breadth of application. It would be great to see future study in this area expand to include uncertainty quantification tools, high-performance computing parallelization approaches, and other physical phenomena.

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