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MODULUS OF TYKHONOV WELL-POSEDNESS

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Abstract

The modulus of Tykhonov well-posedness quantifies the stability and uniqueness of solutions to optimization problems, particularly in the context of constrained non-convex optimization. A problem is Tykhonov well-posed if it has a unique minimizer, and every minimizing sequence converges to this minimizer. The modulus measures the rate at which the objective function grows away from the minimizer, providing a metric for the "degree" of well-posedness. For constrained non-convex problems, the modulus is influenced by the curvature of the objective function and the geometry of the feasible set defined by constraints. It is typically analyzed through conditions like strong convexity or weaker growth conditions near the solution. In non-convex settings, the modulus may only hold locally, and its computation often involves the second-order properties of the objective and constraints, such as the Hessian or constraint qualifications. The modulus is critical for assessing convergence rates of algorithms and robustness to perturbations, with applications in machine learning, control theory, and inverse problems.

Keywords : *Tykhonov well-posedness, Function, Non-convex, Control theory etc.*

Introduction

A problem is Tikhonov well-posed if it has a unique solution (or minimizer) and every sequence of approximate solutions (a minimizing sequence) converges to this solution. This is particularly relevant for minimization problems, where one seeks to minimize a function $f: X \to \boldsymbol{R}$ over a set X, and the goal is to ensure that numerical methods producing approximate solutions reliably converge to the true solution.

Modulus of Tikhonov Well-Posedness

The modulus of Tikhonov well-posedness quantifies how "well-behaved" the convergence of approximate solutions is under perturbations of the problem's data. It provides a way to estimate the stability of the solution with respect to changes in the input or parameters. Specifically:

- It measures the size of the set of approximate solutions to the original problem (or a perturbed version) as the degree of approximation improves.
- For mathematical programming problems with data perturbations, the modulus can be used to derive estimates of the condition number, which reflects how sensitive the solution is to small changes in the input data.

Formally, for a minimization problem $\min x \in X \ f(x)$ the modulus of Tikhonov well-posedness may involve analyzing the behavior of the solution set under perturbations, often expressed through metrics like:

- The rate at which approximate solutions (e.g., ϵ-solutions) converge to the true solution as $\epsilon \to 0$.
- The size of the set of approximate solutions, which can be bounded based on the well-posedness properties.

The modulus of Tykhonov well-posedness is a key concept in optimization, particularly for constrained non-convex problems, quantifying the stability and uniqueness of solutions. Tykhonov well-posedness, introduced by Andrey Tikhonov for ill-posed problems, ensures a problem has a unique minimizer to which all minimizing sequences converge. In non-convex settings, the modulus is often local, relying on assumptions like smoothness, constraint qualifications or positive definiteness of the Lagrangian's Hessian. By providing a quantitative measure of well-posedness, the modulus bridges theoretical insights and practical algorithm design, addressing challenges posed by non-convexity and constraints in optimization.

Since Tykhonov Well-Posedness of (X, f) amounts to the existence of some $x_0 \in$ arg min (X, f) such that

$$f(x_n) \to f(x_0) \text{ implies } d(x_n, x_0) \to 0,$$

It is reasonable to try to find some estimate from below for $f(x)$-$f(x_0)$ in terms of $d(x, x_0)$. This aims to quantitative results about Tykhonov Well-Posedness. Before giving the quantitative result we give the definition of forcing function.

Definition 1.1

A function c : $D \to [0, +\infty)$ is called a forcing function if and only if $0 \in D \subset [0, +\infty)$, c (0) = 0 and $a_n \in D$ such that $c(a_n) \to 0$ implies $a_n \to 0$.

Example 1.2

$c(t) = t^k$, $t \geq 0$, $k > 0$.

Remark 1.3

(D, c) is Tykhonov Well-Posed if c is a forcing function.

We now prove the following theorem for the B-vex Well-Posed optimization problem

Theorem 1.4

Let Y be a non-empty convex subset of X and f be a B-vex function with respect to $b_1(x, y, \lambda)$ with $0 < b_1(x, y, \lambda) < 1$ and (Y, f) be Tykhonov Well-Posed. Then the function c defined by

$$x(t) = \inf \{f(x) - f(x_0) : d(x, x_0) = t\}, t \geq 0$$

where x_0 = arg min (Y, f) is strictly increasing and forcing.

Proof.

Let (Y, f) be Tykhonov Well-Posed with solution x_0.

We want to prove c is forcing.

We see that $c(0) = 0$ and $c(t) \geq 0$ for all t.

Let $t_n \geq 0$ be such that $c(t_n) \to 0$.

Then there exists a sequence $u_n \in Y$ such that $f(u_n) \to f(x_0)$, $d(u_n, x_0)t_n$ for all n.

Then, $\{u_n\}$ is a minimizing sequence.

Therefore, $u_0 \to x_0$.

This implies that $t_n \to 0$.

Thus c is forcing.

Now, we prove that c is strictly increasing.

Let $0 < a < b$ be such that Y contains points u_1, u_2 with $\| u_1 - x_0 \|$ and $\| u_2 - x_0 \| = b$.

Give $x \in Y$ such that $\|x - x_0\| = b$, put

$$z = sx + (1-s)x_0 \in Y$$

where $s = a/b$,

then $\|z - x_0\| = s\|x - x_0\| = a$.

Now, by B-vexity of f

$$f(z) = f(sx + (1-s)x_0)$$

$$\leq sb_1(x,y,\lambda)f(x) + (1 - sb_1(x,y,\lambda))f(x_0)$$

$$f(z) - f(x_0) \leq sb_1(x,y,\lambda)\{f(x) - f(x_0)\}$$

Therefore

$$\leq f(z) - f(x_0)$$

$$\leq sb_1(x,y,\lambda)\inf\{f(x) - f(x_0) : \|x - x_0\| = b, x \in X\}$$

$< c(b)$ (since $b_1 < 1$, $s < 1$, therefore $sb_1 < 1\}$ Therefore c is strictly increasing.

Hence the theorem.

Remark 1.5

The function e defined above is called the modulus of Tykhonov Well-Posedness of (Y, f).

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