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Recreation in Mathematics: Puzzles of Famous Mathematicians

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ABSTRACT
Recreation is a practice where we tend to relax ourselves from our routine lifestyle. It is a way more relaxing and tension free. Recreation in Mathematics is similar to that. Mathematics has always been a branch of research and study which made it difficult for many students those who studied mathematics. Puzzles in other hand solved this problem. They created interest towards the subject as well as the problem. Children and Scholars found it fun to solve the problems and relax from their routine study pattern. In this paper we discuss several puzzles which were found and solved by some Famous Mathematicians.

KEYWORDS: *Amicable numbers, ibn Qorra, Josephus Problem, Tower of Hanoi, Lucas, Ramanujan's Unknown Address.*

History of Puzzles

Puzzle is a recreational activity in mathematics. Mathematics doesn't constrain itself only with research activities. It can also be used to entertain or engage oneself. Puzzle is one such activity. Solving a puzzle has a lot of benefits. Now let us have a look at the history of puzzles.

The mathematical games, puzzles and riddles have a long history going back as far as about 1800 BCE. They can range from simple arithmetic to logic puzzles to complex problems whichare still unsolved today. Mathematical games and puzzles are all throughout the history of mathematics, some of which have led to the study of many areas of math. Early Egyptian Mathwas based largely on puzzle type problems.

Example: The Rhind Papyrus (1850 BCE) "Seven houses contain seven cats. Each cat kills seven mice. Each mouse had eaten seven yearsof grain. Each ear of grain would have produced seven hekats of wheat. What is the total of allof these?"

Solution:

$$7^1 + 7^2 + 7^3 + 7^4 + 7^5 = 7 + 49 + 343 + 2401 + 16807 = 19607.$$

Let us have a look at some early mathematical puzzles.

The Tower of Hanoi

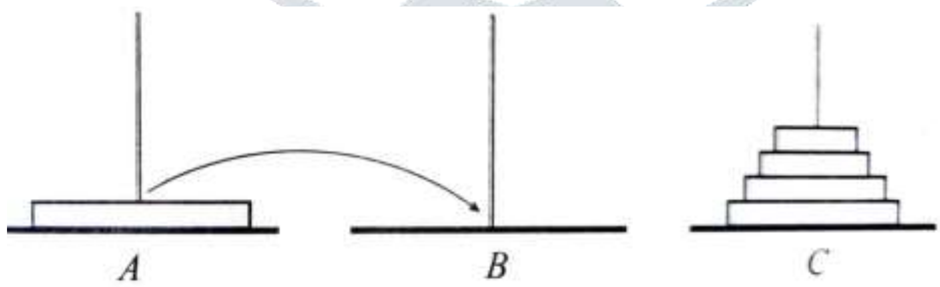
The Towers of Hanoi, also called Tower of Brahma, Lucas' Tower, or simply, the pyramid puzzle, is a mathematical game using three nods and various numbers of colored disks stackedin descending order with the larger disk on the bottom and the smaller disks stacked on top. This creates the iconic cylinder or pyramid shape that the Tower of Hanoi is known for.

The Tower of Hanoi puzzle was first introduced by French mathematician Eduard Lucas in 1883, many myths about its origins have been shared. The most popular one is the mystical theory that it is an homage to an ancient Indian temple in Kashi Vishwanath where Brahmin priests have been acting out an ancient prophecy since the beginning of history (this is where the name Tower of Brahma came from).

The game consists of three pegs standing upright. Upon one of the pegs are a series of rings, decreasing in radius from bottom to top. There are usually from 3 to 8 rings, with the original game possessing 8. The goal is to simply move all the rings from the initial peg to another in the fewest moves possible based on a two simple rules.

1. Only one ring may be moved at a time.
2. A larger ring may not be placed upon a smaller ring.

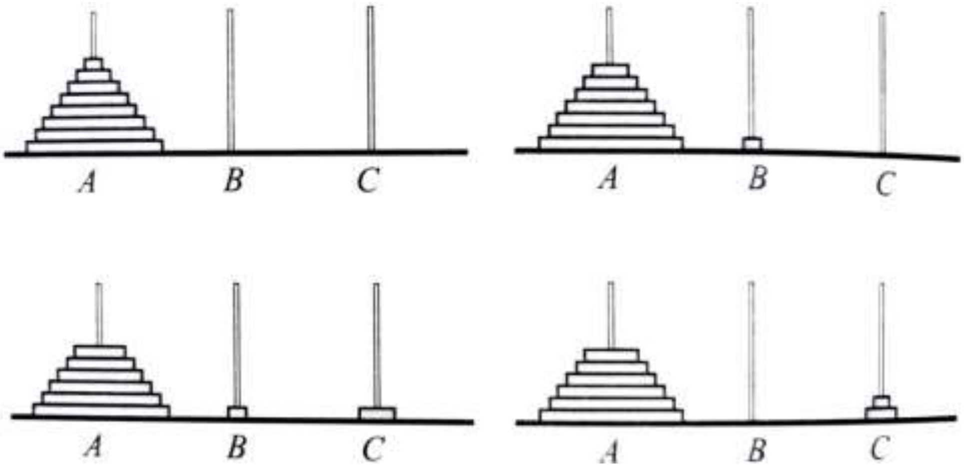
Problem: The tower puzzle consists of three vertical pegs set into a board and a number of disks graded in size, eight disks in the case of Lucas' toy, as we see in Figure below. These disks are initially stacked on one of the pegs so that the largest rests at the bottom of the stack,the next largest in size atop it and so on, ending with the smallest disk placed at the top. A player can shift the disks from one peg to another one at a time, however, no disk may rest upon a disk smaller than itself. The task is to transfer the tower of disks from the peg upon which the disks initially rest to one of the other pegs. How does one accomplish this transfer in the minimum number of moves?



Denote the required minimal number of moves with h_n (h stands for "Hanoi"). It is evident that $h_1 = 1$ and $h_2 = 3$. The first three moves are displayed as in below figure.

In order to transfer the largest disk from peg A to peg B, we must first construct a tower composed of the remaining n – 1 disk on peg C, using the peg B (see as below). The minimal number of moves necessary for this transfer is h_{n-1}. After that, one move is needed moves to transfer n – 1 disks from peg C to peg B by using peg A. Therefore, the required number is given by the recurrence relation

$$h_n = 2h_{n-1} + 1,\ n \geq 2,\ h_1 = 1. \qquad (1.1)$$



For n, n – 1, …, 3, 2 the relation (1.1) gives

$$h_n = 2h_{n-1} + 1 \quad h_{n-1} = 2h_{n-2} + 1$$

$$\vdots$$

$$h_3 = 2h_2 + 1 \quad h_2 = 2h_1 + 1$$

Multiplying the above relations by 1, 2, 2^2, …, 2^{n-2}, respectively and summing the left and right sides of multiplied relations, after cancelling the same terms one obtains

$$h_n = 1 + 2 + 2^2 + \cdots 2^{n-2} + 2^{n-1} = 2^n - 1.$$

Therefore, $2^n - 1$ is the minimum number of moves required to affect the complete transfer of n disks. In the case of Lucas' 1883 toy with n = 8 disks, this number is $2^8 - 1 = 255$. In fact, $2^n - 1$ is really the minimum number of moves if the described algorithm is applied, but we cannot yet claim "mission finished".

Sudoku

Magic squares are known all over the work in mathematical games like Sudoku and Kenken. The history of Sudoku dates back to an 18th century Swiss mathematician's game called 'Latin Squares' and some of the first number puzzles to appear in newspapers were published in France 1895. But the modern game of Sudoku as we recognize it today was invented by Howard Garns, a freelance puzzle inventor from Connersville, Indiana, USA in 1979 when it was published in Dell Pencil Puzzles and World Games magazine.

The puzzle was known as 'Number Place', since it involved placing individual numbers into empty spots on a 9x9 grid. The game first appeared in Japan in 1984 where it was given the name 'Sudoku' which is short for a longer expression in Japanese – 'Suji wa dokushin ni kagiru' – which means 'the digits are limited to one occurrence', Sudoku continues to be highly popularin Japan, where people buy over 600,000 Sudoku magazines per month. The man whoreintroduced Sudoku 'back' to the Western world was a New Zealand judge named Wayne Gould, who was on vacation in Tokyo in March 1997 when he discovered Sudoku in a bookstore. He quickly became a devoted enthusiast of Sudoku and spent the next 6 years developing a computer program that could generate Sudoku puzzles. The objective is to fill a 9x9 grid with digits so that each column, each row and each of the nine 3x3 sub-grids contain all the digits from 1 to 9.

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

The Kakuro Puzzles.

The Kakuro puzzle is the mathematical way of 'crosswords' where the white or empty blocks are filled with numbers 1 to 9 such that they are matched with the sum in the left corner block. To summarize the rules: Kakuro is a puzzle game on a crosswords-like board where digits are used in order to make them sum up to values specified in the 'definition' squares of the board. In addition, inside each sum group, each digit can appear once at most.

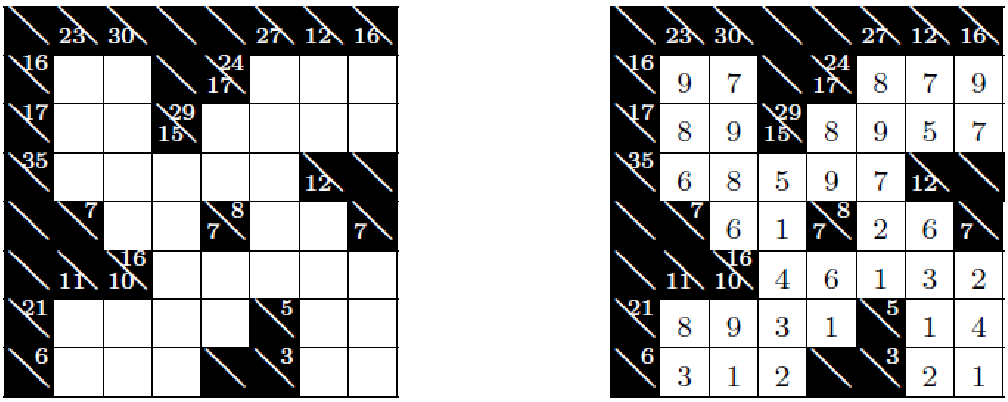


Table 1. Example Problem and Solution

The traditional way to solve a Kakuro puzzle is incremental: by using the existing information on the board, you can find with certainty the value of a specific cell which can take only one possible value. Then that value is filled and the process is repeated until all the board cells havebeen discovered.

In some situations, there's no specific board cell having just one possibility. In those cases, each of the possibilities needs to be explored on its own and eliminated through contradictionsuntil only one course of action remains.

Kakuro Magic Block:

The secret to solving Kakuro puzzles is learning how to use magic blocks – those special situations where only a single combination of numbers can fit into a block of a given length. For example, if 6 is the sum-clue of a block of three squares then the block must consist of the numbers 1+2+3 but not necessarily in this order.

The Kakuro puzzles appear in nearly 100 Japanese magazines and newspapers. Kakuro remained the most popular logic puzzle in Japanese printed press until 1992, when Sudoku took the top spot. In the UK, they first appeared in The Guardian with The Telegraph and the Daily Mail following.

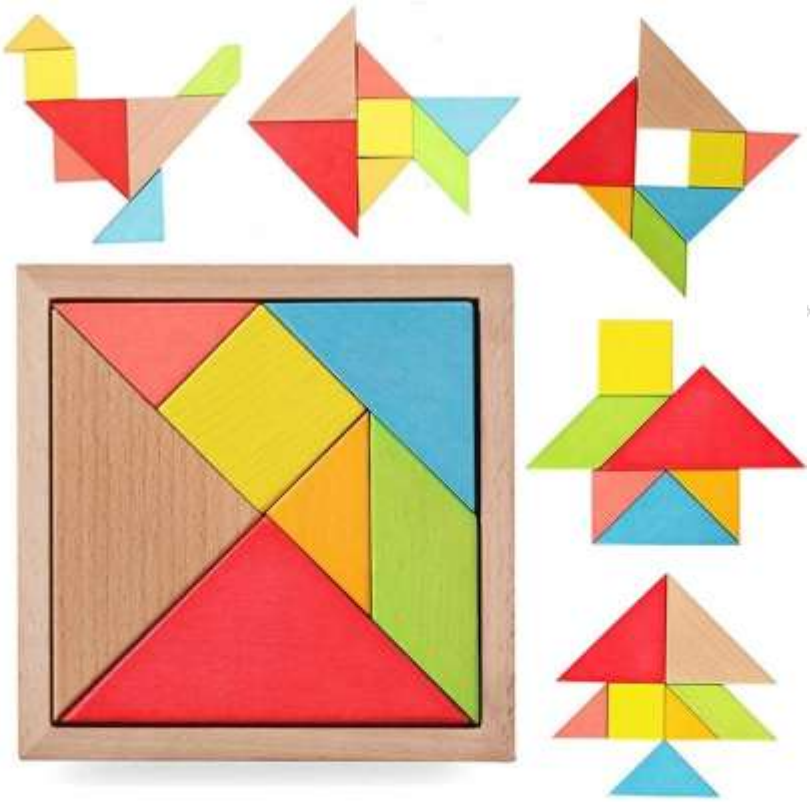
Sum	Block	Combination	Sum	Block	Combination
3	2	1+2	22	6	1+2+3+4+5+7
4	2	1+3	38	6	3+5+6+7+8+9
16	2	7+9	39	6	4+5+6+7+8+9
17	2	8+9	28	7	1+2+3+4+5+6+7
6	3	1+2+3	29	7	1+2+3+4+5+6+8
7	3	1+2+4	41	7	2+4+5+6+7+8+9
23	3	6+8+9	42	7	3+4+5+6+7+8+9
24	3	7+8+9	36	8	1+2+3+4+5+6+7+8
10	4	1+2+3+4	37	8	1+2+3+4+5+6+7+9
11	4	1+2+3+5	38	8	1+2+3+4+5+6+8+9
29	4	5+7+8+9	39	8	1+2+3+4+5+7+8+9
30	4	6+7+8+9	40	8	1+2+3+4+6+7+8+9
15	5	1+2+3+4+5	41	8	1+2+3+5+6+7+8+9
16	5	1+2+3+4+6	42	8	1+2+4+5+6+7+8+9
34	5	4+6+7+8+9	43	8	1+3+4+5+6+7+8+9
35	5	5+6+7+8+9	44	8	2+3+4+5+6+7+8+9
21	6	1+2+3+4+5+6	45	9	1+2+3+4+5+6+7+8+9

Tangrams

A Tangram is a puzzle or a tricky set of seven geometric shapes made up of two small triangles,one medium triangle, two large triangles (total five triangles), a square, and a parallelogram. These triangles are called as tans. When the pieces of a tangram are arranged together they display an amazing variety of forms and personify many numerical and geometric concepts. Tangram pieces are broadly used to solve puzzles. All seven pieces can be fit together to form a square. Observe the following figure that shows a seven-piece tangram.

Why Do We Use Tangrams?

Like building blocks, tangrams can develop problem-solving and logical thinking skills, perceptual reasoning, visual-spatial awareness, creativity, and many mathematical concepts such as congruency, symmetry, area, geometry, and perimeter. They help kids learn geometric terms and develop stronger problem-solving abilities. Children can perform better in tests of basic arithmetic with the help of tangrams. There are over 6,500 shapes and pictures that can be created using a tangram puzzle.



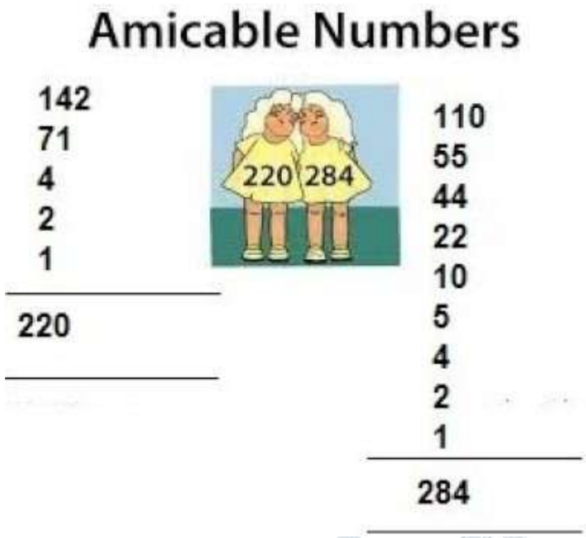
Amicable Numbers

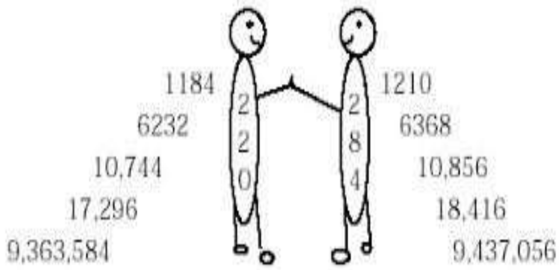
The discovery of amicable or friendly numbers is ascribed to Greek mathematicians. It is said that two numbers are amicable if and only if the sum of the proper divisors (the divisors excluding the number itself) of each of them is equal to the other. Pythagoras is attributed to finding one such pair 220 and 284. Indeed, the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20,
22, 44, 55, 110 and their sum is 284, while the sum of the proper divisors 1, 2, 4, 71, 142 of 284 is just 220. This was the only known pair for some 2000 years. So, this extraordinary
property of the two numbers was the cause of a superstition that a pair of talismans bearing these numbers would ensure a friendship between the wearers.

***Tābit ibn Qorra* (826 – 901)**

The Arabian scientist Tābit ibn Qorra (or Qurra, following V. Katz), famous for his remarkabletranslations of Euclid (Elements), Apolonius, Archimedes, Ptolomy and Theodosius, wrote an elementary algebra, conics and astronomy, but also on fanciful topics such as magic squares and amicable (or friendly) numbers.

Pythagoras set the amicable pair (220, 284) (the only one known in his time) in the context of a discussion of friendship saying that a friend is “one who is the other I such as are 220 and 284.” According to Beiler, there was once an experiment performed by El Madschrity, an Arabof the eleventh century, who tested the erotic effect of the amicable pair (220, 284) by giving one person 220 special cakes to eat, and himself eating 284 at the same time. Unfortunately, ElMadschrity did not report the outcome of this experiment.





Ibn Qorra’s theorem:

For n>1, let $q_n = 3.2^n-1$, $r_n = 9.2^{n-1}-1$. *If* q_{n-1}, q_n *and* r_n *three prime numbers, then* $A = 2^n q_{n-1} q_n$ *and* $B = 2^n r_n$ *are amicable.*

Qorra’s rule for generating amicable pairs was rediscovered by Fermat (1636) and Descartes (1638).

The simplest case n=2 gives primes $q_1 = 5$, $q_2 = 11$, $r_2 = 71$ and the resultant pair of amicable numbers is (220, 284) which was already known to the Greeks. Despite the existence of sucha fruitful formula, no new pair of amicable numbers were discovered unit another Islamic mathematician Kamāl al-Dīn al-Fārisī announced (17,296, 18,416) as a second pair. Three centuries later the great French mathematician Pierre de Fermat rediscovered the same pair. This pair can be obtained from ibn Qorra’s for n=4. The next pair of amicable numbers generated by ibn Qorra’s formula is (9,363,584, 9,437,056) which is obtained for n=7. Let us note that, after a systematic study of amicable numbers, Leonard Euler gave (1747) a list of 30 pairs and later be extended it to 64 pairs (two of which were shown later to be unfriendly). Euler stated the following rule for generating amicable pairs.

If three natural numbers

$$p = 2^m (2^{n-m} + 1) -1,$$

$$q = 2^n (2^{n-m} + 1) -1, \; r = 2^{n+m} (2^{n-m} + 1)^2 -1$$

are all prime for some positive integer m satisfying $1 \leq m \leq n-1$, *then the numbers* $2^n pq$ *and* $2^n r$ *are an amicable pair.*

Euler's rule is a generalization of ibn Qorra's formula (the case m=n-1). The first few pairs (m,n) which satisfy Euler's rule are (m, n) = (1,2), (3,4), (6,7) ……generating the amicable numbers (220, 284), (17,296, 18,416), (9,363,584, 9,437,056) the same ones found by ibn Qorra's formula. Another more powerful, generating formula was discovered in 1972 by W. Borho.

Curiously enough, the above lists and other lists made up to 1866 do not include the relativelysmall pair of amicable numbers 1,184 and 1,210. This pair was discovered by a sixteen-year- old Italian boy, Niccolo Pagamini, not to be confused with the composer and violinist of the same name! This example, as well as many other examples, show that ibn Qorra's , Euler's and Borho's formula do not generate all pairs of amicable numbers.

In recent years, the number of known amicable numbers has grown explosively thanks to powerful computer machines. For example, there 1,427 amicable pairs less than 10^{10} and 5001less than 3.06 x 10^{11}.

Josephus Problem:

n people numbered 1 to n are disposed around a circle. If one starts with person 1 and eliminates every second remaining person, determine the survivor's number, J(n) (J stands for Josephus).

The mentioned authors have derived in a simple but remarkable recurrence relation that defines J(n) for arbitrary n. In what follows we give a short adaptation of their treatment of the Josephus problem. Let us distinguish between the even and odd case of n. If there are 2n people at the start, then after the first trip around the circle only the odd numbers are left and 3 will be the next to eliminate. This is just like the starting position, but without n eliminated persons and a new numeration where each person's number is doubled and decreased by 1. Therefore,

$$J(2n) = 2J(n) - 1, \text{ for } n \geq 1. \qquad (4.1)$$

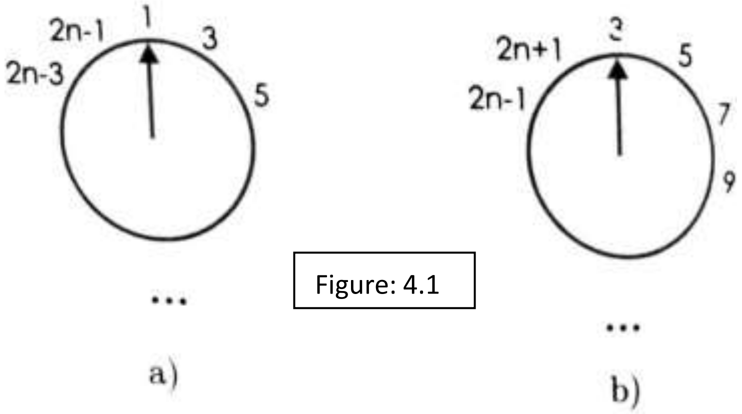


Figure: 4.1

Now let us assume that we have 2n + 1 people originally. After the first pass around the circle,the persons numbered 2, 4, 6, ..., 2n and 1 (in this order) are expected, leaving again persons with odd numbers, excepting number 1 (see Figure (4.1(b))). This resembles the original situation with n people, but in this case their numbers are doubled and increased by 1.

$$\text{Thus, } J(2n + 1) = 2J(n) + 1, \text{ for } n \geq 1 \quad (4.2)$$

Combining (4.1) and (4.2) and taking into account the obvious case J (1) = 1, we obtain a recurrence relation that defines J in all cases:

$$J(1) = 1;$$
$$J(2n) = 2J(n) - 1, \text{ for } n \geq 1; \qquad (4.3)$$
$$J(2n+1) = 2J(n) + 1, \text{ for } n \geq 1.$$

To find a general explicit formula for J (n), we first for Table 4.1 of small values using (4.3).

n	1	2 3	4 5 6 7	8 9 10 11 12 13 14 15	16
J (n)	1	1 3	1 3 5 7	1 3 5 7 9 11 13 15	1

We have grouped the entries of J (n) for n from 2^m to $2^{m+1} - 1$ (m = 0, 1, 2, 3), indicated by the vertical lines in the table. We observe from the table that J (n) is always 1 at the beginning of a group (for n = 2^m) and it increases by 2 within a group. Representing n in the form n = 2^m + k, where 2^m is the largest power of 2 not exceeding n, the solution to the recurrence seems tobe

$$(2^m + k) = 2k + 1, for\ m \geq 0\ and\ 0 \leq k < 2^m \qquad (4.4)$$

We note that the remainder k = n – 2^m satisfies $0 \leq k < 2^{m+1} - 2^m$ if $2^m \leq n < 2^{m+1}$.

We will prove the formula (4.4) by induction. For m =0 it must be k = 0 (following the above bounds). Then from (4.4) we have J (1) = 1, which is true. Let m >0 and 2^m + k = 2n +1, then k is odd. Assume that for some m>0 the following is valid: J (2^m + k) = 2k +1.

According to this hypothesis and (4.3), we find

$$(2^m + k) = 2(2^{m-1} + (k-1) / 2) + 1 = 2(2(k-1) / 2 + 1) + 1 = 2k + 1,$$

which completes the proof by induction. In a similar way we derive the proof in the even case,when 2^m + k = 2n. Therefore, formula (4.4) is stated.

To demonstrate solution (4.4), let us calculate J (101). Since 101 = 2^6 + 37 (that is, k = 37), wehave J (101) = 2. 37 +1 = 75.

The authors gave the following interesting solution to J (n) using representations in the binarysystem. If the binary expansion of n is

$$n = (b_m b_{m-1} \dots b_1 b_0)_2, \text{that is, } n = b_m . 2^m + b_{m-1} . 2^{m-1} + \dots + b_1 . 2 + b_0 \text{ then}$$

$$J(n) = (b_{m-1} b_{m-2} \dots b_1 b_0 b_m)_2.$$

Therefore, J (n) is obtained from n by a one-bit cyclic shift to the left. For example, if n = 101 = $(1100101)_2$ then

$$J(101) = J((1100101)_2) = (1001011)_2,$$

which gives

$$J(101) = 64 + 8 + 2 + 1 = 75.$$

We end this essay on the Josephus problem with the remark that the Mathematics package *Combinatorica* can stimulate the "Josephus process" by the command **Inverse Permutation[Josephus[n, m]].** The outcome is the ordered list of men who are consecutively eliminated from the circle consisting of n men, when every mth man is eliminated. For example, n =41 and n =3 in the original Josephus problem (problem 4). The command **Inverse Permutation[Josephus[41, 3]]** gives the ordered list

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39,
1, 5, 10, 14, 19, 23, 28, 32, 37, 41, 7, 13, 20, 26,
34, 40, 8, 17, 29, 38, 11, 25, 2, 22, 4, 35, **16, 31**.

The last two numbers (in bold) represent beneficial places for Josephus and his friend.

Seven Towns and One-way Roads

Problem:

There are 7 towns in a country, each of them connected to the other by a two-way road. Can one reconstruct all of these roads as one-way roads so that for any two specified towns it is always possible to reach each town in one step from some third town?

By drawing a digraph with 7 vertices, we can employ arrows whose orientations indicate the particular direction of one-way roads and arrayed in such a way so that for any specified pair of towns, there is a third town from which you can drive directly to the other to. The solution is shown below.

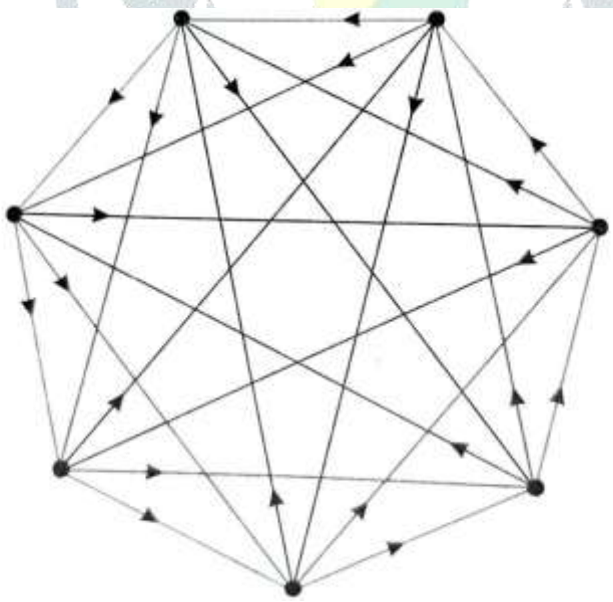


Figure: Graphing the one-way road problem

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