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PROPERTIES OF SEMI MULTIPLICATION IN ALMOST DISTRIBUTIVE LATTICES

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Abstract :
__________ In this paper, mainly, we have introduced the concept of Semi Multiplication in Almost Distributive Lattice (ADL). We have proved important properties of Semi Multiplication in Almost Distributive Lattice.

Key words: Semi Multiplication, Almost Distributive Lattice.

I. INTRODUCTION

In 1981, Swamy, U. M., and Rao, G. C. made significant advancements in the field of Boolean algebra with their work on Almost Distributive Lattices (ADL) [6]. Although ADLs lack full commutativity for ∨ (logical OR), ∧ (logical AND) commutativity, and right distributivity of ∨ over ∧, they still adhere to most conditions of a distributive lattice. Moreover, the class of ADLs introduced new possibilities by extending concepts from distributive lattices through the formation of a distributive lattice using its principal ideals.
The concept of residuation, dating back to Dedikind, found its way into ideal theory and saw abstract exploration in multiplicative structures by Ward [7, 8]. Ward, M., and Dilworth, R.P., delved into residuated lattices in [9, 10]. A Residuated Almost Distributive Lattice was defined in [3] by Rao, G.C., and Raju, S.S., who also introduced the ideas of Residuation and Multiplication in an Almost Distributive Lattice. Building on these ideas, this paper introduces the innovative notion of Semi Multiplication in ADL.
In section 2.1 of the paper, we revisited the foundational definition of an Almost Distributive Lattice (ADL) and explored essential characteristics outlined by Rao, G.C. [2], and Swamy, U.M., and Rao, G.C.[6].
In section 3.1, we have introduced the concept of semi multiplication in an ADL L and derived important properties of semi multiplication in an ADL L.

2.1 PRELIMINARIES

In the context of this study, several key definitions and results serve as the foundation for our research. Among these, the fundamental definition of an Almost Distributive Lattice (ADL) is paramount.

An Almost Distributive Lattice, abbreviated as ADL, is a specialized form of Boolean algebra characterized by specific properties. While ADLs do not necessarily exhibit full commutativity for logical OR (∨) or logical AND (∧) operations, and lack right distributivity of ∨ over ∧, they still adhere to most conditions typical of distributive lattices. In essence, an ADL satisfies nearly all distributive lattice conditions except for these specific exceptions.

This unique structure, as defined by Swamy, U. M., and Rao, G. C. [6], has paved the way for extensive exploration and generalization within the realm of Boolean algebra. The distinctive properties of ADLs make them a crucial area of study, serving as a foundation upon which subsequent research in this field has been built.

Throughout this work, we will leverage and build upon these established definitions and results, utilizing them as the basis for our investigations. By delving into the intricacies of ADLs and their related concepts, we aim to contribute significantly to the ongoing discourse in this domain

Definition 2.1.1[2]: An **Almost Distributive Lattice**(ADL) is an algebra (L, ∨, ∧) of type (2, 2) satisfying the following conditions

I. $(a\vee b)\wedge c = (a\wedge c)\vee(b\wedge c)$
II. $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$
III. $(a\vee b)\wedge b = b$
IV. $(a\vee b)\wedge a = a$
V. $a\vee(a\wedge b) = a$, $\forall$ a, b, c ϵ L.

According to the definition given above, each distributive lattice is an ADL.

An Almost Distributive Lattice (ADL) of type (2, 2, 0) is said to have an element 0 if there is an element $0 \in L$ such that for every $l \in L$, $0 \wedge l = 0$; this is also known as an ADL with 0.
In any ADL L, the following results hold.
Thusforth, an ADL (L, ∨, ∧, 0) will be denoted by L.

Theorem 2.1.1 [2]: If for any three elements l, m, n ∈ P, then we have the following :

(1) $0 \vee l = l$, $l \wedge 0 = 0$ and $l \wedge l = l = l \vee l$
(2) $(l \wedge m) \vee m = m$, $l \vee (m \wedge l) = l$ and $l \wedge (l \vee m) = l$
(3) $l \wedge m = l \Leftrightarrow l \vee m = m$ and $l \wedge m = m \Leftrightarrow l \vee m = l$
(4) $l \wedge m = l \wedge m$ and $l \vee m = m \vee l$ whenever $l \leq m$
(5) $l \wedge m \leq m$ and $l \leq l \vee m$
(6) ∧ is associative in P
(7) $(l \vee m) \wedge n = (m \vee l) \wedge n$
(8) $l \wedge m \wedge n = m \wedge l \wedge n$
(9) $l \vee (m \vee l) = l \vee m$
(10) $l \wedge m = 0 \Leftrightarrow m \wedge l = 0$.

Definition 2.1.2[2]: Let L be an ADL and m be an element in L, then m is called **a maximal element** of L, if it satisfying any one of the following.

I. $m \wedge x = x$ for any x ϵ L
II. $m \vee x = m$ for any x ϵ L

3.1. Properties of Semi Multiplication in Almost Distributive Lattices

In this section, we have introduced the concept of Semi Multiplication in an Almost Distributive Lattice (ADL) L and prove important properties of Semi Multiplication in an ADL L.

First we have given the following definition.

Definition 3.1.1: Let L be an ADL and m be a maximal element in L. A binary operator '. ' in an ADL L is called **Semi Multiplication** on L, if for any a, b, c ϵ L then the following conditions are satisfied.

(M1) $[(a.b)\wedge c]\wedge m=[(b.a)\wedge c]\wedge m$

(M2) $[(a.b).c]\wedge m=[a.(b.c)]\wedge m$

(M3) $[(a.m)\wedge m]=a\wedge m$

(M4) $[a.(b\vee c)]\wedge m=[(a.b)\vee(a.c)]\wedge m$.

Lemma 3.1.1

Let L be an ADL with a maximal element 'm' and " . " is a semi-multiplication in L. For a ϵ L and for every maximal element m_1 of L, we have

$(a.m_1)\wedge(a.m)\wedge a=(a.m)\wedge a$

Proof:

Let $a \in L$ and m_1 be a maximal element of L, then

$[(a.m_1) \vee (a.m)] \wedge a = [(a .m_1) \vee (a. m)] \wedge m \wedge a$

$= [a.(m_1 \vee m)] \wedge a$ (by M4)

$= (a.m_1)\wedge a$ ($m_1 \vee m = m_1$)

⇨ $[(a.m_1)\vee(a.m)\wedge a = (a.m_1)\wedge a$

Since $x\vee y = x$ ⇔ $x\wedge y = y$ for any $x, y \in L$

$(a.m_1) \wedge (a.m)\wedge a = (a.m)\wedge a$

Theorem 3. 1. 1

Let L be an ADL with maximal element m and ". " is a binary operator on L satisfying the condition M1 for any $a , b , c , x \in L$

$a\wedge(x.b) = (x.b)\wedge c$ ⇔ $a\wedge[(b.x)\wedge c] = (b.x) \wedge c$

Proof:

Let $a, b, c, x \in L$

$a\wedge[(x.b)\wedge c] = (x.b)\wedge c$

$a\wedge[(x.b)\wedge c]\wedge m = (x.b)\wedge c\wedge m$

⇔ $a\wedge[(x.b)\wedge c]\wedge m =(x. b)\wedge c\wedge m$

⇔ $a\wedge[(b.x)\wedge c]\wedge m = (b. x)\wedge c\wedge m$ (by M1 of definition 3.1.1)

⇔ $a\wedge(b.x)\wedge c = (b. x)\wedge c$

Lemma 3. 1. 2

Let L be an ADL, for any $a, b \in L$, $a\wedge b = b$ and $b\wedge a = a$ ⇔ $a\wedge x = b\wedge x$, for any $x\in L$

Proof:

Let $a, b, x \in L$

Suppose $a\wedge b = b$ and $b\wedge a = a$

Then, $a\wedge b\wedge x = b\wedge x$

⇨ $a\wedge x\wedge b\wedge x = b\wedge x$

⇨ $b\wedge x\wedge a\wedge x = b\wedge x$

⇨ $b\wedge x \leq a\wedge x$ → (1)

Similarly, we prove that $a\wedge x \leq b\wedge x$ → (2)

From (1) and (2), we get that

$a \wedge x = b\wedge x.$

Now suppose that $a \wedge x = b\wedge x$ for any $x\in L$ →(3)

Put $x = b$ in (3), we get that $a\wedge b = b$

and put $x = a$ in (3), we get that $b\wedge a =a$.

Lemma 3. 1. 3

Let L be an ADL with a maximal element m and '.' is a semi-multiplication on L. For $a \in L$ and m_1 be any maximal element of L then we have

$(a.m)\wedge m = a\wedge m$ => $(a.m_1)\wedge x = a\wedge x$ for any $x \in L$

Proof:

Let $a, x \in L$ and m_1 be any maximal element of L.

Suppose that $(a.m)\wedge m = a\wedge m$.

Then $(a.m)\wedge a = a$ and $a\wedge(a.m) = a.m$

Now $(a.m_1)\wedge a = (a. m_1)\wedge(a.m)\wedge a$ (Since $a = (a.m)\wedge a$)

$= (a.m)\wedge a$

= a

$(a.m_1) \wedge a = a.$

Now, $a \wedge (a.m_1) = (a.m) \wedge a \wedge (a.m1)$ (by $a = (a.m) \wedge a$)

⇨ $a \wedge (a.m_1) = a \wedge (a.m) \wedge (a.m_1)$

⇨ $a \wedge (a.m_1) \wedge m = (a.m) \wedge (a.m_1 \wedge m$ (by $a \wedge (a.m) = a.m$)

⇨ $a \wedge (a.m_1) \wedge m = (a.m_1) \wedge m$ (By Lemma 3.1.1)

⇨ $a \wedge (a.m_1) = (a.m_1)$

Hence, by Theorem 3.1.1, we get that

$(a.m_1) \wedge x = a \wedge x.$

Theorem 3. 1. 2

Let L be an ADL with a maximal element m and ' . ' is a semi-multiplication on L, then for any a , b, c, d, e ϵ L We have the following:

i. $a \wedge (a.b) = a.b$ and $b \wedge (a.b) = a.b$

ii. $a \wedge b = b$ => $(c.a) \wedge (c.b) = c.b$ and $(a.c) \wedge (b.c) = b.c$

iii. $d \wedge [(a.b).c] = (a.b).c \Leftrightarrow d \wedge [a.(b.c)] = a.(b.c)$

iv. $(a.c) \wedge (b.c) \wedge [(a \wedge b).c) = (a \wedge b).c$

v. $d \wedge (a.c) \wedge (b.c) = (a.c) \wedge (b.c)$ => $d \wedge [(a \wedge b).c] = (a \wedge).c$

vi. $a \wedge [(a.c) \vee (b.c)] = (a.c) \wedge (b.c) \wedge d \Leftrightarrow a \wedge [(a \vee b).c] \wedge d = [(a \vee b).c] \wedge d$

vii. $If\ a \wedge m = b \wedge m$ => $(a.c) \wedge m = (b.c) \wedge m$

viii. $If\ a \wedge m = b \wedge m$ and $c \wedge m = d \wedge m$ => $(a.c) \wedge e \wedge m = (b.d) \wedge e \wedge m.$

Proof:

(i) $(a \wedge m) \vee [(a.b) \wedge m] = [(a.m) \wedge m] \vee [)a.b) \wedge m]$ ($By\ M3$ of definition 3.1.1)

$=[(a.m) \vee (a.b)] \wedge m$

$= [a.(m \vee b)] \wedge m$ ($By\ M4$ of definition 3.1.1)

$= (a.m) \wedge m$ ($Since\ m \vee b = m$)

$= a \wedge m$ ($by\ M3$ of definition 3.1.1)

⇨ $a \wedge m \wedge (a.b) \wedge m = (a.b) \wedge m$

⇨ $a \wedge (a.b) = a.b$

Interchanging a and b in the above, we get that

$$b \wedge (b.a) = b.a$$

⇨ $b \wedge (b.a) \wedge m = (b.a) \wedge m$

⇨ $b \wedge (a.b) \wedge m = (a.b) \wedge m (byM1)$

⇨ $b \wedge (a.b) = a.b.$

(ii)

Suppose $a \wedge b = b. then\ \ a \vee b = a.$

$$Now\ [(c.a) \wedge m] \vee [(c.b) \wedge m$$

$$= [(c.a) \vee (c.b)] \wedge m$$

$$= [c.(a \vee b)] \wedge m \qquad (by\ M4\ of\ definition\ 3.1.1)$$

$$= (c.a) \wedge m$$

$$Therfore, (c.a) \wedge m \wedge (c.b) \wedge m = (c.b) \wedge m\ and\ hence (c.a) \wedge (c.b) = c.b$$

Similarly, we can prove that, $(a.c) \wedge (b.c) = b.c.$

(iii) $Suppose\ that\ d \wedge [(a.b).c] = (a.b).c$

$\Leftrightarrow d \wedge [(a.b).c] \wedge m = [(a.b).c] \wedge m$

$\Leftrightarrow d \wedge [a.(b.c)] \wedge m = [a.(b.c)] \wedge m$ ($by\ M2$ of definition 3.1.1)

$\Leftrightarrow d \wedge [a.(b.c)] = a.(b.c).$

(iv) By (i), above, we have $a \wedge a \wedge b = a \wedge b$

⇨ $(a.c) \wedge [(a \wedge b).c] = (a \wedge b).c$

Thus, $(a.c)\wedge(b.c)\wedge[(a\wedge b).c] = (a\wedge b).c$

(v) Assume that $d\wedge(a.c)\wedge(b.c) = (a.c)\wedge(b.c)$

$\Rightarrow \quad d\wedge(a.c)\wedge(b.c)\wedge[(a\wedge b).c] = (a.c)\wedge(b.c)\wedge[(a\wedge b).c$

$\Rightarrow \quad d\wedge[(a\wedge b).c] = (a\wedge b).c\big(by(4)\big)$

(vi) *Suppose* $q\wedge[(a.c)\vee(b.c)]\wedge d = [(a.c)\vee(b.c)]\wedge d$

$\Leftrightarrow q\wedge[(a.c)v(b.c)]\wedge d\wedge m = [(a.c)v(b.c)]\wedge d\wedge m$

$\Leftrightarrow q \wedge[(a.c)\wedge d]\vee[(b.c)\wedge d]\wedge m = [(a.c)\wedge d]\vee[(b.c)\wedge d]\wedge m$

$\Leftrightarrow q\wedge[(c.a)\wedge d]\vee[(b.c)\wedge d]\wedge m = [(c.a)\wedge d]\vee[(c.b)\wedge d]\wedge m$ ($by\ M1$ of definition 3.1.1)

$\Leftrightarrow q\wedge[(c.a)\vee(c.b)]\wedge d\wedge m = [(c.a)\vee(c.b)]\wedge d\wedge m$

$\Leftrightarrow q\wedge[c.(a\vee b)]\wedge d\wedge m = [c.(a\vee b)]\wedge d\wedge m$ ($by\ M4$ of definition 3.1.1)

$\Leftrightarrow q\wedge[(a\vee b).c]\wedge d\wedge m = [(a\vee b).c]\wedge d\wedge m$ ($by\ M1$of definition 3.1.1)

$\Leftrightarrow q\wedge[(a\vee b).c]\wedge d = [(a\vee b).c]\wedge d$

(vii) *Suppose* $a\wedge m = b\wedge m.$

Then $a\vee b = a$ and $b\vee a = b.$

$Now\ (a.c)\wedge d\wedge m = [(a\vee b).c]\wedge d\wedge m$ ($Since\ a = a\vee b$)

$= [c.(a\vee b)]\wedge d\wedge m$ ($by\ M1$ of definition 3.1.1)

$= [(c.a)\vee(c.b)]\wedge d\wedge m$ ($by\ M4$ of definition 3.1.1)

$= [(c.b)\vee(c.a)]\wedge d\wedge m$

$= (c.b)\wedge d\wedge m$ ($Since\ b\vee a = b$)

$= (b.c)\wedge d\wedge m$ ($by\ M1$ of definition 3.1.1)

$Hence, (a.c)\wedge d\wedge m = (b.c)\wedge d\wedge m, for\ any\ a, b, c, d \in L.$

(viii) *Suppose* $a\wedge m = b\wedge m$ and $c\wedge m = d\wedge m.$

By (vii), we get that $(a.c)\wedge e\wedge m = (b.c)\wedge e\wedge m$ and $(b.c)\wedge e\wedge m = (b.d)\wedge e\wedge m$

Thus $(a.c)\wedge e\wedge m = (b.d)\wedge e\wedge m.$

We conclude this paper with the following Theorem.

Theorem 3.1.3:

Let L be an ADL with a maximal element m, "." is a semi-multiplication on L and a, b ϵ L such that $a\wedge b = b$, *then* $a^n\wedge b^n = b^n$, *for any* $n \in Z^+$.

Proof:

Let a, b ϵ L be such that $a\wedge b = b$

This result is proved by induction on n

Clearly, the result is true for n = 1.

Assume that $a^k \wedge b^k = b^k$, *for some* $k \in z^+$

Then $(a^k.b)\wedge(b^k.b) = b^k.b$ (by Theorem 3.1.2 condition (ii))

That is, $(a^k.b^k)\wedge b^{k+1} = b^{k+1}$ → (1)

Now, $a^{k+1}\wedge b^{k+1} = (a^k.a)\wedge b^{k+1}$

$= [a^k.(a \vee b)]\wedge m \wedge b^{k+1}$ (since $a = a \vee b$)

$= [a^{k+1}\vee (a^k.b)]\wedge m \wedge b^{k+1}$ ($by\ M4\ of definition$ 3.1.1)

$= [a^{k+1} \vee (a^k.b)] \wedge b^{k+1}$

$= (a^{k+1}\wedge b^{k+1}) \vee [(a^k.b)b^{k+1}]$

$= (a^{k+1}\wedge b^{k+1}) \vee b^{k+1}$ (by (1) above)

$= b^{k+1}.$

Hence, the result.

CONCLUSION

In conclusion, we present the culmination of our research in the form of the importance of Semi Multiplication in Almost Distributive Lattice (ADL). Through rigorous analysis and meticulous proofs, we have established properties of paramount importance within this specialized domain of Boolean algebra.

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