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# Review of Graph Theory And It's Application

**Gyanvendra Pratap Singh[1]*and Simran Gupta[2] †**

Department of Mathematics and Statistics

Deen Dayal Upadhyaya Gorakhpur University

Gorakhpur-273009, (U.P.), India

Abstract

Graph theory is widely used to prove many mathematical theorems and models. This paper present the various applications and techniques of graph theory to solve problems in different fields of science and technology in addition to mathematics. A graph can be used to represent almost any physical situation involving discrete objects and a relationship among them. This abstract provides a concise overview of graph theory's foundational principles, including graph types(such as directed, undirected, weighted, and unweighted graphs), basics terminologies( vertices, edges, paths, cycles), and essential theorems (e.g, Euler's theorem Hamiltonian cycles). Moreover, it highlights practical applications of graph theory, such as shortest path algorithms (e.g., Dijkstra's algorithm), network flow optimization, and graph color-ing problems. By unraveling the intricacies of graph theory, this abstract aims to foster a deeper understanding of its role in shaping modern computational paradigms and problem-solving methodologies. These fields include website designing, chem-istry, biology, computer science, software engineering and operations research.

Keywords:- Eulerian graph, Hamiltonian graph, Chromatic number, Tree, Kur-atowski's theorem

## 1. Introduction

Graph theory is a foundational branch of discrete mathematics that deals with the study of graphs, which are mathematical structures consisting of vertices ( or nodes)

connected by edges ( or links). In graph theory, graphs are used to model relation-ships between objects, with vertices representing entities and edges representing the connections or interactions between them. The study of graph theory involves ex-ploring various properties and characteristics of graphs, such as connectivity, paths, cycles, and coloring. It also investigated different types of graphs, including directed graphs ( where edges have associated weights pr costs). Graph theory provides a powerful framework for analyzing and solving problems in diverse areas, including computer science, operations research, biology, telecommunication, and social sci-ences.

It offers a range of algorithms and techniques for solving problems related to network connectivity, routing, optimization, and modelling. Overall, graph the-ory serves as a fundamental tool in discrete mathematics, offering insights into the structure and behaviour of complex systems represented by graphs and enabling the development of solutions to real-world problems across various domains. Graphs are used to represent networks of communication, data organisation and find shortest path in road or a network . In Google maps, various location are represented as vertices or nodes and the roads are represented as edges and graph theory is used to find the shortest path between two nodes. Graph theory in mathematics refers to the study of graphs , which are the main subject of discrete mathematics. In any graph, the edges are used to connect the vertices. We can use the application of linear graphs not only in discrete mathematics but we can also use it in the field of Biology, computer science, Linguistics, Physics, Chemistry, etc. GPS (Global po-sitioning system) is the best real-life example of graph structure because GPS has used to track the path or to know about the road's direction.

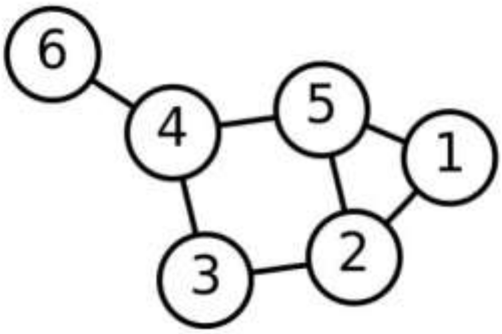
# 2. History

History of graph theory in discrete mathematics traces back to the 18th century when the mathematicians like Leonard Euler explored the famous Seven Bridges of konisberg problem in 1736. Euler's solution to this problem laid the foundation for graph theory by introducing the concept of a graph as a mathematical abstraction to represent and analyze relationships between objects. However, it wasn't until the 19th century that more formal developments in graph theory emerged. In 1847, Gustav Kirchhoff introduced the concept of a graph in the context of electrical cir-cuits, laying the groundwork for the study of network theory. Later in the century, Arthur cayley pioneered the study of trees ( a special type of graph) and intro-duced the concept of graph isomorphism. The late 19th and early 20th centuries saw significant contributions to graph theory from mathematicians such as William Hamilton cycles, and Percy MacMahon, who studied graph enumeration problems. During the mid-20th century, graph theory experienced a surge of interest and devel-opment, particularly with the advent of computers. Mathematicians and computer scientist like Claude Shannon, Paul Erdos, and Edges Dijkstra made notable con-tributions to graph theory, developing algorithms and techniques for solving graph related problems. In the later half of the 20th century and into the 21st century,

graph theory continued to evolve rapidly, with applications spreading across various fields, including computer science, operations research, biology, telecommunications and social sciences. The development of new algorithm, theories, and applications has made graph theory an indispensable tool in modern mathematics and science. The history of graph theory states it was introduced by the famous Swiss mathem-atics named Leonard Euler, to solve many mathematics problems by constructing graphs based an given data or a set of points. The graphical representation shows different types of data in the form of bar graphs, frequency tables, line graphs, circle graphs, line plots, etc. Graph theory started in 1735 with the problem of Konisberg bridge. Euler studied the problem of Konisberg bridge and constructed a structure to solve the problem; this structure is called as Eulerian graph. The details complete graph and bipartite graph is given by A.F Mobious. In 1840 , Kuratowski proved that they are planer by means of recreational problems. The concept of tree was given by Gustav Kirchhoff in 1845, which are used in the calculation of currents in electrical networks or circuits, Thomas Gutherie found the four colours problem in 1852. William Hamilton and P. Kirkman discussed the cycle an polyhydra then invented the concept named Hamiltonian graph. In 1913, H. Dudeney mentioned a puzzle problem. In 1878, Sylvester introduced the term "graph" .In 1941, Ramsey worked on the concept called collarations which lead to the identification of another branch of graph theory called extremel graph theory. The four colour problem was solved using computer by Heinnch in 1969.

# 3. Preliminaries

Graph- A graph is denoted as G(V,E) graph consisting of two set vertices 'V' and edges 'E'. In Mathematics, a graph is a pictorial representation of any data in an organised manner. The graph shows the relationship between variable quantities. In a graph theory, the graph represents the set of objects, that are related in some sense to each other. The objects are basically mathematical concepts, expressed by vertices or nodes and the relation between the pair of nodes, are expressed by edge.

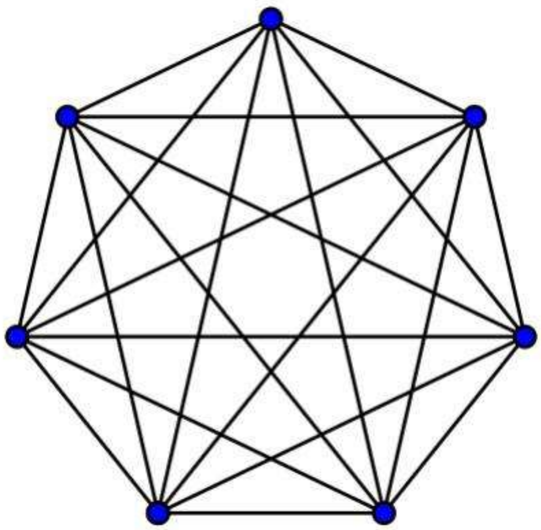



## 3.1 Basic Terminology

- [Trivial Graph] A graph consisting only one vertex and no edge.
- [Null Graph] A graph consisting n vertices and no edge.
- [Directed Graph] : A graph consist the direction of edges then this is called di directed graph.
- [Simple Graph] A graph does not contain any self loop and multiedge.
- [Multigraph]A graph does not contain any self loop but contain multiedge is called multigraph.
- [Isolated Vertex Pendant Vertex] A vertex having 0 is called isolated Vertex and a vertex having degree 1 is called pendant vertex.
- [VFinite and Infinite graph] A graph with a finite number of vertices as well as edges is called finite graph otherwise it is an infinite graph.
- [Pseudo Graph] A graph contain both self loop and multiedge is called pseudo Graph.
- [Undirected Graph] A graph which is not directed then it is called undirected graph.
- [Self loop in a Graph] if edge having the same vertex as both its end vertices is called self loop.
- [Proper edge] An edge which is not self loop is called proper edge.
- [Multi edge] A collection of two or more edges having identically end point.

## 3.2 SOME IMPORTANT GRAPHS

1)[Complete Graph] A simple connected graph is said to be complete if each ver-tex is connected to every other vertex



2) [Regular Graph] A graph G is said to be regular if every vertex has the same degree. If degree of each vertex of graph G is K, then it is called k-regular graph.

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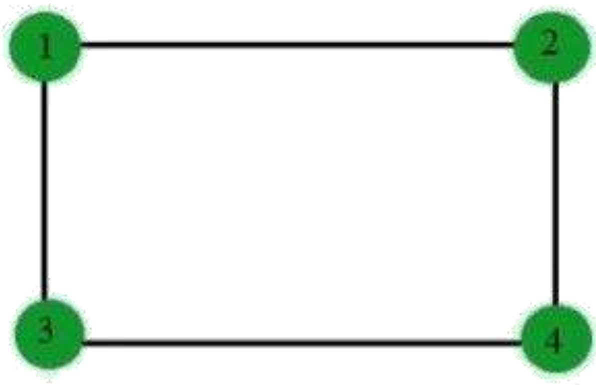
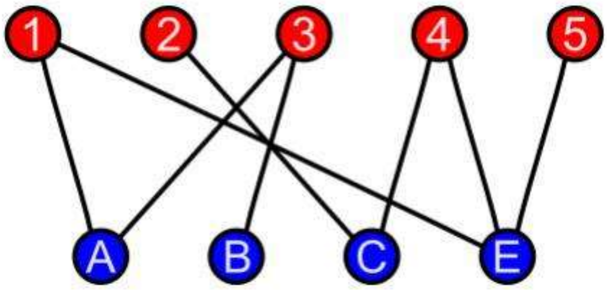
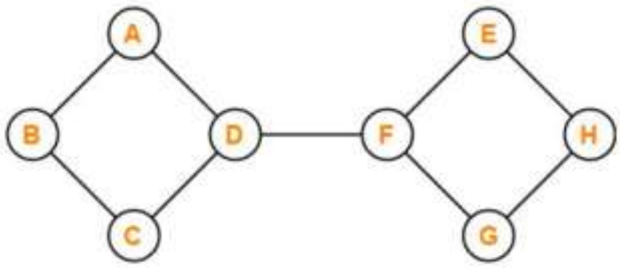



3) [Bigraph ( or Bipartite)] If the vertex set V of a graph G can be partitioned into two non- empty disjoint subsets X and Y in such a way that edge of G has one end in X and one end in Y. Then G is called bipartite.




4)[CONNECTED GRAPH] An undirected graph is said to be connected if there is a path between every two vertices.

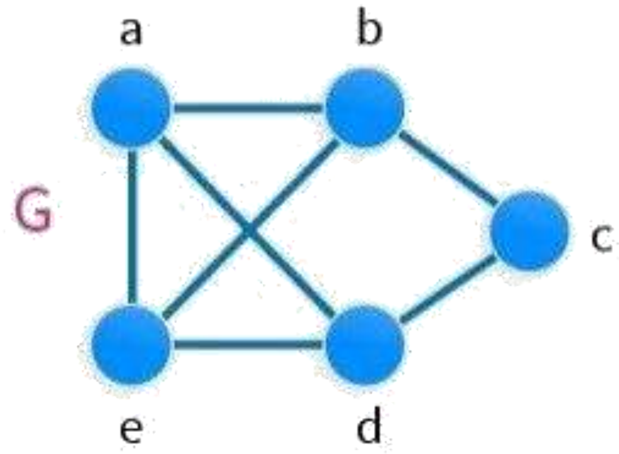



Example of Connected Graph

*Remarks- : If a graph is connected then it will not bipartite

1)[ Complete Bipartite Graph] : If every vertex in X is disjoint is every vertex in Y, then it is called a complete bipartite graph. If X and Y condition m n vertices then this graph is denoted by vn.

2) [Subgraph] Let G(V, E) be a graph. Let V' be a subset of V and let E' be a subset of E whose end point belong to V'. Then G(V' , E') is a graph and called a subgraph of G(V, E)




## 4. Decomposition of graph

A graph is said to be decomposed into two subgraph G1 G2 if G1G2 = G G1G2 null graph.

## 5. Complement of graph

The complement of a graph G is defined as a simple Graph with the same vertex set as G and where two Vertices u and v are adjacent only when they are not adjacent in G.

## Planare graph

A graph which can be drawn in the plane so that it's Edges do not cross is called planare.

## Rooted tree

A rooted tree is a tree in which one vertex is root.

## Binary tree

A binary tree is defined as a tree in which there is exactly one vertex of degree two and each of remaining vertices is of degree one or three and vertex of degree two is serves as a root.

## Pendent vertex in tree

A vertex of degree one is called pendent vertex of tree.

## Path length of tree

path length of a tree is defined as the sum of edges from the root of all pendent vertices.

## Spanning tree

If G is any connected graph a spanning tree in G is a subgroup T of G, which is a tree.

## Eulerian path

A path in a graph is said to be an eulerian path if it traverses each edge in the graph once and only once.

## Eulerian circuit

A circuit in a graph is said to be an eulerian circuit if it traverses each edge in the graph once and only once.

## Eulerian grap

A connected graph which contains an eulerian circuit is called eulerian graph.

## Hamiltonian path

A path which contains every vertex of a graph G exactly once is called Hamilto-nian graph.

## Hamiltonian circuit

A circuit that passes through each of the vertices in a group G exactly one except the starting vertex and end vertex is called Hamiltonian circuit.

## Hamiltonian graph

A connected graph which contain Hamiltonian circuit is called Hamiltonian graph.

## Weighed graph

A graph is called weighed graph if a non-negative integer w(e) associate to each edge and this w(e) is a weight of corresponding edge.

## Chromatic Number

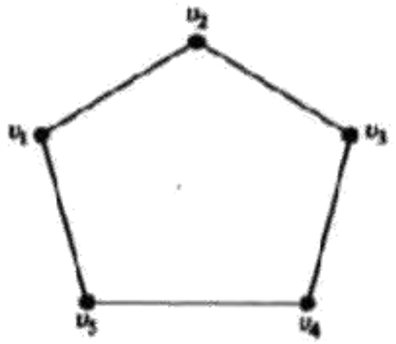
The least number of colors required for coloring of a graph G is called it's chro-matic number.

# 6. KURATOWSKI'S THEOREM

Kuratowski's Theorem is a fundamental result in graph theory that characterizes planar graphs. It states that a graph is planar if and only if it does not contain a subgraph homeomorphic to either the complete graph k5 ( the complete graph on five vertices) or the complete bipartite graph k3,3( the complete bipartite graph with two sets of three vertices each). In simpler terms, a graph can be drawn on a plane without any edges crossing each other if and only if it does not contain a subgraph that looks like K5 or K3,3. Planarity is such a fundamental property, it is clearly of importance to know which graphs are planar and which are not. We have already noted that, in particular, Ks and K3•3 are nonplanar and that any proper subgraph of either of these graphs is planar. A remarkably simple characterisation of planar graphs was given by Kuratowski (1930).

Theorem : The complete graph of five vertices is nonplanar.

Proof: Let the five vertices in the complete graph be named v1, v2, v3, v4, and v5. A complete graph, is a simple graph in which every vertex is joined to every other vertex by means of an edge. We must have a circuit going from v1 to v2 to v3 to v4 to v5 to v1 – that is, a pentagon. This pentagon must divide the plane of the paper into two regions, one inside and the other outside (Jordan curve theorem). Since vertex v1 is to be connected to v3 by means of an edge, this edge may be drawn inside or outside the pentagon. Suppose that we choose to draw a line from v1 to v3 inside the pentagon. (If we choose outside, we end up with the same argu-ment.) Now we have to draw an edge from v2 to v4 and another one from v2 to v5. Since neither of these edges can be drawn inside the pentagon without crossing over the edge already drawn, we draw both these edges outside the pentagon. The edge connecting v3 and v5 cannot be drawn outside the pentagon without crossing the edge between v2 and v4. Therefore, v3 and v5 have to be connected with an edge inside the pentagon. Now we have yet to draw an edge between v1 and v4. This edge cannot be placed inside or outside the pentagon without a crossover. Thus the graph cannot be embedded in a plane. A complete graph with five vertices is the first of the two graphs of Kuratowski. The second graph of Kuratowski is a regular† connected graph with six vertices and nine edges, where it is fairly easy to see that the graphs are isomorphic.

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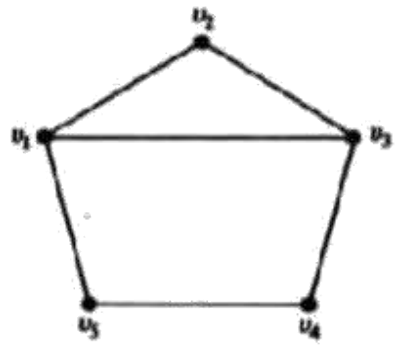


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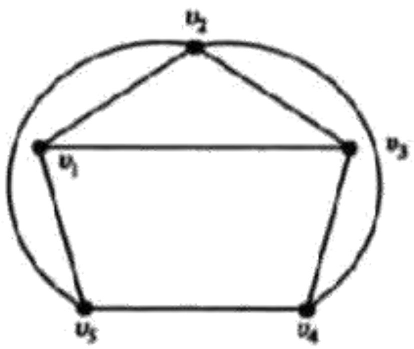


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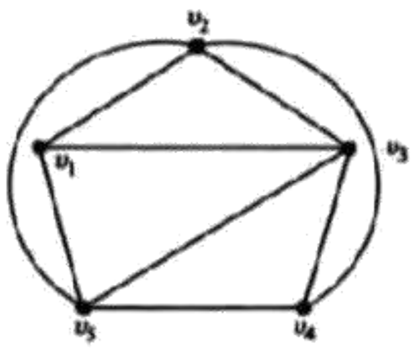


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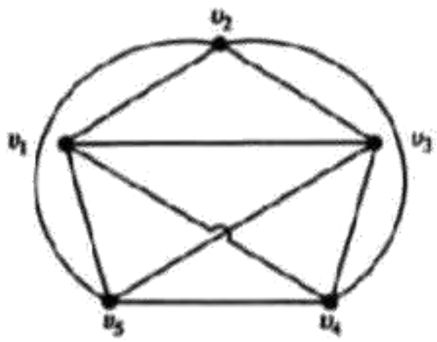


(e)

Theorem: Kuratowski's second graph is also nonplanar

Proof: Several properties common to the two graphs of Kuratowski. These are

1. Both are regular graphs.
2. Both are nonplanar.
3. Removal of one edge or a vertex makes each a planar graph.
4. Kuratowski's first graph is the nonplanar graph with the smallest number of vertices, and Kuratowski's second graph is the nonplanar graph with the Smallest number of edges. Thus both are the simplest nonplanar graphs.

In the literature, Kuratowski's first graph is usually denoted by K5 and the Second graph by K3,3—letter K being for Kuratowski.

## 7. Application of graph theory

Graph theory, a branch of discrete mathematics, has numerous application in vari-ous fields.

### (A) Computer Network –

Graphs are used to represent networks of communication, Data organisation, computational devices, the flow of computation etc. One practical example is the link structure of a website could be represented by a directed graph

### (B) Networks :

Graph theory is extensively used in the study of networks, such as social Net-works, transportation networks , communication networks and computer

### (C) Networks.

It helps in analysing connectivity, identifying critical nodes, optimising routes, And understanding networks resilience.

**(D) Computer science:** Graph theory plays a vital role in com-puter science, particularly in the design and Analysis of algorithms. It is used in data structures Like adjacent lists for representing graphs. Algorithms like Dijk-stra's algorithm for shortest Paths, Prim's algorithm for minimum Spanning trees, and algorithms for graph Traversal ( e.g.,depth-first search and breadthFirst search) are all based on graph theory.

**(E) Circuit Design:** In electrical engineering, graph theory is ap-plied to analyze and Design circuits.Graph models help in representing connections between Components and analysing properties like voltage distribution, current flow and circuit efficiency.

## (F) Operations Research :

Graphs are used to model optimization problems like the traveling salesman problem, where the objective is to find the shortest routes that visits a set of cities exactly once and returns to the origin City.

## (G) Biology :

Graph theory is applied in bioinformatics for modelling molecular structures, proteinprotein interactions, and genetics networks.

## (H) Chemistry :

Graph theory is used to model and analyze molecular structures, chemical reac-tions, and chemical bonding patterns.

## (I) Transportation :

Graph theory is used to model transportation networks and analyze traffic flow, optimising routes, and minimising congestion.

## (J) Operation Management :

Graph theory is applied in project management to model project activities, de-pendencies, and critical paths.

## (K) Epidemiology :

Graph theory plays a crucial role in modelling the spread of diseases. Nodes represent individuals, and edges represent contacts or interactions between them. Epidemiologists use graph algorithms to simulate dis-ease transmission, identify key influencers, and develop strategies for diseases control.

## (L) Power Grids :

Graph theory is employed in modelling and analyzing electrical power grids. Nodes represent power stations, substations, and consumers, while edges represent transmission lines. Graph algorithms help in optimizing power flow identifying vul-nerabilities, and designing resilient power grid systems.

## (M) Game Theory :

Graph theory is applied in graph theory to model strategic interactions between players in various games, including social dilemmas, voting systems, and economic competitions. Graph algorithms help analyze equilibrium strategies, coalition form-ations, and game dynamics.

## (N) Genetics :

Graph theory is used in genetics for analyzing genetic networks, genome se-quencing, and evolutionary relationship between species. Graph algorithms help in

ntifying genetic patterns, predicting gene functions, and understanding genetic diseases.

## (O) Geography

Graph theory is used in geographic information systems (GIS) for spatial ana-lysis, route planning, and mapping. It helps in analyzing spatial relationships and optimizing geographic data .

# 8. Acknowledgment

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# 9.Conclusion

Graph theory plays a fundamental role in discrete mathematics, offering powerful tools and concepts for analyzing and solving a wide range of problems. Through its study, mathematics and researchers gain insights into the structure, connectivity, and properties of networks and discrete structures. The applications if graph theory extend across numerous fields, including computer science, biology, telecommunica-tions, geography, and social sciences, making it a versatile and indispensable area of study. With its rich theoretical foundations and practical applications, graph theory continues to inspire new discoveries and innovations, shaping our understanding of complex systems and networks in the digital age and beyond.

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