JETIR.ORG ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue
JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)
An International Scholarly Open Access, Peer-reviewed, Refereed Journal

"Study of Signal Processing with Fractional Calculus"

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I. ABSTRACT

The signal processing has experienced extraordinary growth and research over the past decades. Fractional calculus comes as path of advancement of Signal processing with its property tooperate non-integer order of differentiation and integration. We will be discussing the theoretical approaches, problems and applications of signal processing with Fractional Calculus within modern technologies under this paper.

Keywords:- Fractional Calculus, Signal Processing, Fractional Derivatives, Fractional Integrals, Image Processing, Discrete Fractional Calculus, MRI, Biomedical Applications

II. INTRODUCTION

There has been remarkable advancement in signal processing and it plays a significant role in different fields, e.g. image processing, Internet of Things, communication or health diagnosis etc. Traditional signal processing does not rely on modern complex and mixed signals, although it does use integer order calculus. For this reason, researchers are coming up with alternative methods using Fractional Calculus to give a new life to modern signal processing. We will be discussing the theoretical approaches, problems and applications of signal processing with Fractional Calculus within modern technologies under this paper.

III. INTRODUCTION TO FRACTIONAL CALCULUS

Fractional calculus is a branch of mathematics that studies the differentiation and integration of fractional order .Fractional calculus is a generalization of conventional mathematics, using the same concepts and tools but applying them more consistently than they used to. As we know that non-integer order of the operators can define the dynamical behavior of the process over a huge time and frequency scale.

Before we get into the fractional calculus, let's talk about some key functions that are connected to the fractional calculus.

A. Gamma Function:- The gamma function is a generalization of factorials of realnumbers[1]. It can be defined as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \qquad x \in R^+$$

The Gamma function has the following properties

$$\Gamma(x+1) = x\Gamma(x) \qquad x \in R^+$$

$$\Gamma(x) = (x-1)! \qquad x \in N$$

The gamma function is utilized in a wide range of applications in Signal Processing, such as Fourier domain analysis, Filter design, Wavelet transformation, etc.

B. Beta Function:- The beta function is a generalized form of the definite integral [1]. It can be defined as

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt \qquad x,y \in x \in R^+$$

Beta function in terms of Gamma function

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \qquad x,y \in x \in R^+$$

Let's talk about some basic definitions of fractional integrals and fractional derivatives.

According to Riemann -Liouville the Fractional integral of f of order v is [1]

$$cD_x^{-v} f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t) dt \qquad v > 0$$

Proof:-

Let's consider a n[th] order differential equations with initials conditions

$$(x) = f(x) \qquad (1)$$

$$(c) = y'(c) = \cdots \dots \dots = y^{n-1}(c) = 0$$

By Cauchy function

$$H(x,t) = \frac{(x-t)^{n-1}}{(n-1)!}$$

Now by using (1)

$$y(x) = \int_c^x H(x,t) f(t) dt$$

$$y(x) = \int_c^x \frac{(x-t)^{n-1}}{(n-1)!} f(t) dt$$

According to Riemann-Liouville the Fractional derivative of f of order v is

$$D^u f(x) = D^n[D^{-v} f(x)] \qquad v = n - u, \quad 0 < v < 1$$

Some properties of fractional derivative [2]:

1. Linearity:- The fractional derivative possess the linearity properties i.e fractional derivative of linear combination of function are equal to the linear combination of their derivative.

$$D^u [f(t) + g(t)] = D^u(f(t) + D^u(g(t))$$

2. Causality:- The fractional derivative has causal property that mean if $t = z \in$R $and\ f(t) = 0\ for\ t < 0\ \ th\ \ for\ t < 0 \qquad D^u f(t) = 0$.

3. Time reversal:- If $f(t) = g(-t)$ then we have

$$D^u\,(-t) = (-1)\;g(T)\;for\;T = -t$$

4. Time shift:- Derivative Operator is invariant with shift operator i.e

$$D^u\,(t - a) = D^u\,(T)\;for\;T = t - a$$

Fractional Calculus is utilized in wide range of applications such as Mortgage Problem In Economy, Time decay problem, Electromagnetism, Robotics, Thermal Engineering and Signal Processing.

IV. FRACTIONAL DIFFERENTIAL EQUATIONS

Fractional differential equations form the basis of Signal modeling. There are variety of differential equations available with their respective solutions and explanations, that have their various application in the Signal Processing like time series analysis of signal, noise modeling, image processing, Biomedical signal processing and Fractal antennas and arrays etc.[3] Generally, Laplace Transform is employed in Signal Normal Processing. In Signal Processing normally we solve fractional differential equation using Laplace Transform.

Let us consider a differential equation that represents the fractional diffusion behavior of signal.

$$0D_t^{\alpha}u(x,t) = D_t^{\beta}u(x,t)$$

Caputo fractional derivative, on the left-hand side, represents the time expansion of the signal. It is used for the memory effects and long-range dependencies. The term on the right-hand side, “Riesz fractional,” represents the diffusion of the signal. The above formula is used to model the signal where both the temporal and the spatial behavior of the signal are required.

I. FRACTIONAL FOURIER TRANSFORM

Fractional Fourier Transform (FrFT) is another important term used in Signal processing. It is mainly used in time and frequency domain analysis and non-stationary behavior of signal. The applications of FrFT include image processing, waveform generation, chirp signal analysis, signal compression, Encryption etc.[4]

FrFT of Signal $x(t)$ can be defined as

$$X_{\alpha}(u) = [x(t)] = \int_{-\infty}^{\infty} x(t)K_{\alpha}(t,u)dt$$

Where α is the rotational angle of FrFT and (t, u) is a kernel function of Frft and p is orderof FrFT [3].

II. SOME IMPORTANT METHOD OF APPROXIMATION IN FRACTIONAL CALCULUS

A wide range of methods and numerical approaches are available for continuous and discrete approximations of fractional order elements [5].

1. Continuous time approximation
    a. Continuous fraction expansion
    b. Matsuda' Method
    c. Carson's method

2. Discrete time approximation
    a. Foh linear interpolation
    b. Zoh-zero order hold
    c. Tustin bilinear approximation

III. APPLICATION OF FRACTIONAL CALCULUS IN SIGNAL PROCESSING

A. Non stationary signal spectral analysis

In non-stationary signal the measurements of signal changes with the time. The Short- Time Fourier Transform (STFT) and Fractional Fourier Transform (FRFT) are two techniques used to analyze this type of signal. The STFT analyze the signal by applying the Fourier transform to short segments of signal using moving window and shows the time-frequency variation with the spectrogram. On the other hand, the FRFT is a generalized version of the fractional Fourier Transform, capable of handling signals withdifferent time-frequency characteristics. Additionally, the Short-Time Fractional Fourier Transform (STFRFT) combines the benefits of STFT and FRFT, segmenting signals with a window and applying FRFT to each segment to capture waveform nuances and time- frequency variations of non-stationary signals [6].

B. Biomedical Applications

i. Fractals, Chaos, and Nonlinear Systems: Mandelbrot's work in Fractals, chaos theory and nonlinear systems led to the application of fractional calculus to biomedical applications, particularly to capture complexity in phenomena such as biological signal spectra.

ii. Membrane Biophysics and Viscoelasticity: Initially, fractional calculus was used to solve membrane biophysical problems and to describe the power law dynamics of current flow and stress strain relationships using fractional orders differential equations.

iii. Physiological Modeling: Fractional calculus is commonly used in linear differential equations-based physiological modeling. These models successfully explain a variety of complex systems, including action potential propagation and blood oxygenation, as well as insulin secretion.

iv. Biological Signal Analysis: Electrocardiograms (ECGs), electromagnetism (EMGs), and electroencephalogram (EEGs) all display spectra that deviate from standard models. These unusual signal behaviors led to the development of fractional-order systems models with fractional poles and fractional zeros.

v. Multiscale Modeling: Using fractional calculus, we can create multi-scale models that link events at the molecular level (like ion transport and gas diffusion) with organ-level activities (like blood flow and muscle tension). This helps us bridge the gap between the two levels, bringing together the complexities and behaviors of different parts of the body.

vi. Anatomical and Histological Complexity: Multiscale models can incorporate intricate anatomical and histological details, leading to complex structures with multiple components. Fractional calculus aids in dealing with the resulting complexity.

vii. Empirical Approaches: In contrast, multi-scale models can be characterized by empirical methods with a whole-system description, such as linear models and non-linear models, as well as deterministic and stochastic model models. In addition, Chaotic or Fraktical measures (e.g. Fraktical dimensions and LYAPUNOV exponents) capture important characteristics of observed behavior.

viii. Model Characterization: The fractional calculus method is placed in the LTIC (Linear Time Invariant Causal) system model, where the fractional differential operators are placed between the traditional integer-order operators.

ix. Neural System Description: Fractional order models are frequently utilized to capture the behavior of neural systems, both sensory and motor. For example, Anastasio's vestibular-occulomotor system has a Laplace domain equation, where the Laplace domain is s^a (1 < a ≤ 1). This choice of s^a behavior indicates a requirement to understand and control underlying biological, physical or chemicalmechanisms.

x. Distributed Relaxation Processes: Fractional dynamics models are used to explainthe large dynamic range of sensory adaptation. The previous interpretations of non-linear springs and transmission lines, as well as the Gaussian distribution of exponential rate constant, were insufficient to provide a complete explanation[7].

C. IMAGE PROCESSING:- Fractional differentiation has gained significant prominence across various research domains, including image and signal processing. In the realm of image processing, fractional calculus introduces an important path for filtering and edge detection, offering a novel approach to elevate image quality. By extending derivative and integration concepts to non-integer orders , fractional calculus facilitates a nice analysis of image data[8].

D. Noise denoising:- Recent years have witnessed the integration of fractional calculus into image denoising methods, leading to innovative solutions. By utilizing fractional differential theory, researchers have achieved substantial advancements in signal processing, edge detection, and image enhancement. Fractional differential operatorshave demonstrated superior signal-to-noise ratios compared to integer derivatives, providing enhanced insights into complex signals . Additionally, fractional spline wavelets, harnessing the memory properties of fractional calculus, have proven invaluable in texture singularity checks and image fusion, surpassing the capabilities of integer calculus [9].

E. **Image Fusion:** Fractional calculus-based method provides new path for image fusion, where multiple images are combined to produce a single composite image with enhanced features and reduced noise. This is particularly beneficial in applications such as remote sensing and medical imaging, where the merging of information from different sources is crucial.

F. **Pattern Recognition:** Fractional calculus gives a new approach on pattern recognition tasks. By capturing details and new variations in signals, fractional calculus can contribute to more accurate and robust pattern recognition algorithms, especially in cases where traditional methods struggle to distinguish narrow differences.

G. **Speech and Audio Processing:** The complex and complicated nature of speech and audio signals benefits from the application of fractional calculus. It facilitates the extraction of new acoustic features and patterns, contributing to improved speech recognition, audio classification, and music analysis.

H. **Data Compression**: Fractional calculus can enhance data compression techniques by preserving essential signal features during compression processes. This is useful in scenarios where efficient data storage and transmission are essential, while retaining the integrity of the signal content.

I. **Sensor Networks:** Fractional calculus plays a role in optimizing sensor networks, particularly in scenarios where signals from various sensors must be combined and analyzed. Its integration enhances the accuracy of data fusion and enables morecomplicated sensor network designs.

IV. CHALLENGES WHILE USING FRACTIONAL CALCULUS FOR SIGNAL PROCESSING

i. **Limited Software Tools**: While the field of fractional calculus is expanding day by day, there is still a shortage of specialized software tools and libraries that provide user-friendly implementations of fractional calculus algorithms. Researchers and practitioners often need to develop their own implementations. Which increase the cost.

ii. **Limited Standardization**: Fractional calculus don’t have same level of standardization and globally accepted conventions as traditional calculus. This can lead to variability in terminology, notation, and methods across different applications and research areas[10].

iii. **Interpretability and Physical Meaning**: The interpretation and physical meaning of fractional derivatives and integrals might not be as inbuilt as their integer counterparts. This can cause challenges in explaining results and making meaningful connections to real-world phenomena.

iv. **Overfitting:** Overfitting, the phenomenon where a model captures noise rather than true signal patterns, can be increased when using fractional calculus due to its ability to capture fine details. Proper regularization techniques are necessary to improve this challenge.

v. **Complexity and Computational Cost:** The mathematical complexity of fractional calculus can lead to complex algorithms and computations. Implementing these algorithms efficiently and accurately can be computationally demanding, requiring specialized techniques andresources. Which can lead to increase in computational cost.

vi. **Boundary Conditions:** Defining appropriate boundary conditions for fractional calculus operations can be challenging, particularly when dealing with signals that exhibit complex behaviors at their boundaries. This is especially relevant in scenarios like image processingor time series analysis.

vii. **Parameter Selection**: The order of fractional differentiation or integration is a crucial parameter that affects the behavior of the processed signal. Selecting the appropriate order is very important and requires domain knowledge and experimentation.

V. REAL WORLD EXAMPLES

i. **Viscoelastic Anomalous Diffusion**: Complex liquids, such as those encountered in biological cells or polymeric materials, exhibit a unique viscoelastic behavior—displaying characteristics of both elasticity and viscosity. The diffusion of particles within these complicated liquids displays an interested phenomenon known as anomalous diffusion. Anomalous diffusion refers to the behavior where the mean squared displacement of particlesdoes not conform to the standard linear relationship seen in normal diffusion. Instead, it follows a distinct pattern: sub diffusion for values of α between 0 and 1, indicating restrained particle movement, and super diffusion for values of α greater than 1, signifying enhanced particle mobility.

ii. **Ubiquitous Fractional Order Memory System**: Fractional order systems offer a newperspective on memory within dynamic systems. While traditional integer-order systems depend only on the present state for future behavior, fractional order systems exhibit long memory they are influenced by their entire history. This memory characteristic is a defining feature of various fractional order systems, finding applications in diverse fields such as viscoelastic material behavior and quantum mechanics.

iii. **Fractional calculus in Image Processing:** Fractional calculus is a mathematical discipline that deals with the binding of derivatives and integrals in real or complex orders. It has become increasingly popular due to its wide range of applications in science and engineering. Of particular interest is its role in the solution of integral, differential, or integral-differential problems. This versatility is of great value in image processing as well as in various other seemingly disparate domains. The use of fractional calculus in image processing offers a number of powerful tools that help to improve problem-solving abilities and fine-grained analysis.

iv. **Long-Term management for Discrete Fractional Systems**: Discrete fractional calculus provides a customized solution to inherent problems in engineering contexts that are characterized by time or space structures that are discrete, such as digital images or economic series, or signals. Unlike continuous fractional calculus, which requires numerical discretization, discrete fractional calculus directly addresses the memory effect and numerical errors. By considering the discrete nature of the problem, this approach effectively improves potential inaccuracies, making it a straightforward and effective method for controlling systems with long memory.

v. **Fractional Derivative Models in MRI**: Anomaly diffusion is a common in observation of diffusion weighted MRI of many different materials including brain tissue. This departure from traditional Gaussian diffusion patterns leads to the utilization of fractional derivative models. Within this context, the signal intensity decay, represented by a stretched exponential function, provides insight into anomalous diffusion. The parameter α ($0 < \alpha < 1$) characterizes the degree of this anomalous behavior. By using fractional calculus, these models capture the complex diffusion patterns in a more accurate and descriptive manner.

vi. **Chloride Ion Anomalous Diffusion in Concrete**: Investigating the transport mechanisms of chloride ions in concrete structures reveals complexities beyond the realm of traditional Fick's law of diffusion. Concrete's porous and anisotropic nature, coupled with continuous cement binder hydration, gives rise to anomalous diffusion behaviors. These unique properties necessitate novel approaches for modeling chloride ion movement within concrete. Consequently, researchers turn to concepts from fractional calculus to provide a more nuanced and accurate description of these intricate transport processes.

vii. **Anomalous Dielectric Properties**: Anomalous dielectric properties, observed in materials with disordered structures, have posed a challenge to conventional models. Beyond the simplistic Debye model, the Havriliak–Negami (HN) model presents a more intricate framework to describe the complex interactions between polarization and electric fields. By introducing additional parameters, the HN model captures the complexities of anomalous dielectric responses in disordered materials, contributing to a deeper understanding of these phenomena.

viii. **Fractional Derivative Model for Shape Memory Polymers:**
Shape memory polymers (SMPs) have gather significant attention due to their distinctive ability to memorize shapes and respond to external impulses. Since SMP behavior is temperature-dependent and highly sensitive to thermal fluctuations, fractional derivative models emerge as valuable tools for understanding and predicting their dynamic responses. By using fractional calculus, researchers can craft models that reflect the new interactions between SMP properties and external impulses, enhancing accuracy in various applications from industry to medicine.[11]

ix. **Economic Processes with Memory:**

Economic processes, often perceived as intricate systems influenced by a multitude of factors, have been examined with a focus on memory effects. In the domain of fractional calculus, concepts like fractional differencing and integrating provide valuable insights into economic behaviors. These fractional approaches surpass the traditional discrete-time analysis, offering connections to the broader framework of fractional calculus [12].

VI. CONCLUSION

The study of signal processing with fractional calculus leads us to a exciting world of enhanced understanding and manipulation of complex signals. Fractional calculus provides us new tools that go beyond traditional math, making it easier to work with signals that shows weird or irregular behaviour. By considering the history of these signals, we can build better models and methods to analyze them accurately. An interesting concept we have seen is "anomalous diffusion," where the usual patterns of how particles move are changed. This idea fits perfectly with fractional calculus. It helps us study signals that don't follow the normal rules of how thingsspread out over time. We can use special math called "fractional derivative models" to understand these signals better, giving us insight into their hidden behavior. As technology progresses, we are facing signals that behave in more complicated behaviour. As we keep exploring this field, we will find even more applications of this. This will create new methods, new approaches, and a deeper understanding of the signals that surround us.

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