JETIR.ORG ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue
JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)
An International Scholarly Open Access, Peer-reviewed, Refereed Journal

A brief review on function of bounded variation

Gyanvendra Pratap Singh*, Shilpi Ojha †

Department of Mathematics and Statistics,

Deen Dayal Upadhyaya Gorakhpur University,

Gorakhpur- 273009 ,(U.P.) – INDIA

Abstract : In this paper we mainly work on " Function of bounded variation " . The function of bounded variation in one variable was first introduced by Camille Jordan in 1881. Camille Jordan was a French mathematician known for his work in analysis.

In particular, he studied the properties of function that has finite variation over a given interval. The related topics are Riemann-Stieljes integral, absolute continuity , monotonicity, boundedness ,arc length etc. Also we discuss some algebraic properties of bounded variation and some theorem on related topic of bounded variation.

***Keywords*- Bounded variation, continuity , monotonicity ,partitions , refinement**.

1 Introduction

In this paper explore the function of bounded variation and other related topic which are intituively related with function of bounded variation.

We start our paper by defining the variation of a function and what is meaning of a function to be bounded variation. After this, we shall discuss the algebraic properties of function of bounded variation. We discuss monotonicity of a function on [a, b], which relate class of BV − f unction.

Next, we define "Absolute continuity" and nice property to show that all absolute continuous function are class of $BV - function$ but all continuous function of bounded variation are not absolute continuous by given counter example.

We relate arc length and bounded variation. We discuss that a curve which have finite arc length is said to be ractifiable curve and all curve which have finite arc length are class of $BV - function.$

Lastly, we define Riemann-Stieltjes integral and discuss some important theorem to see how it related with function of bounded variation.

2 Preliminaries.

Before we start our discussion on function of bounded variation we must define some basic definitions, lemma and theorems etc.

2.1 Definitions.

2.1.1 Lower bound, upper bound and bounded set.

Let A be a non-empty set of real numbers.

1. Set A is said to be bounded above if ∃ a real number M ∈ R such that M ≥ x , ∀x ∈ A then the real number M is called set an upper bound of A and the collection of all such M is called set of all upper bound.

2. Set A is said to be bounded below if ∃ a real number m ∈ R such that m ≤ x , ∀x ∈ A then the real number m is called lower bound of A and the set of all such m is called set of all lower bound of A.

3. The set A is said to be bounded if it is bounded above as well as bounded below. From above two points we can write if ∃ a no. p such that |x| ≤ p.

2.1.2 Supremum and Infimum of a set.

Let A ⊂ R be bounded above then A has an upper bound and the set of all upper bound of A has a least element is called supremum of A or lowest upper bound. Similarly ,if A is bounded below then it has a lower bound and the set of all lower bound of A has a greatest element is called infimum of a set A or greatest lower bound .

2.1.3 Partition.

Let [a, b] be a closed and bounded interval then a set P ={x_0, x_1, x_2, ..., x_n} such that a = x_0 < x_1 < x_2 < ... < x_n = b is said to be partition on [a, b].The family of all partition on [a, b] is denoted by P[a, b] then if we take a partition P then we can write P ∈ P [a, b]

3 Function of bounded variation.

3.0.1 Variation of a function.

Variation of a function tells to how the output of a function will change as the input values changes. It tells us whether the function is increasing or decreasing or constant. A variation of a function that measures the total changes and fluctuation of a function over an interval.

Let [a, b] be a closed and bounded interval and a function $f: [a, b] \to \mathbb{R}$ and let a partition $P = \{x_0, x_1, x_2, \dots, x_n\}$ where a = x_0 < x_1 < x_2 < ... < x_n = b then the sum

$V(P, f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \dots + |f(x_n) - f(x_{n-1})|$ is said to be variation on a given interval.

Also, we can write

$$V(P, f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \quad , \forall\ i = 1,2,3, \dots, n$$

3.0.2 Bounded Variation.

For different partitions $V(P, f)$ be a non-negative real number and if the set $\{V(P, f) : \forall P \in P[a, b]\} \leq \infty$ i.e finitely exist then f is said to be function of bounded variation or in short $BV - function$.

Mathematically, we write

$$V(P,f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < \infty \quad , \forall \text{ i} = 1,2,3, \dots, \text{n}$$

3.0.3 Total Variation.

$\sup\{V(P,f): P \in P[a,b]\} = V_a^b(f)$

$V_a^b(f)$ is called total variation of a function f.

3.0.4 Examples

$$\boldsymbol{f(x) = x\,; x \in [a,b]}$$

Here , variation of a function

$$V(P,f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \quad , \forall \text{ i} = 1,2,3, \dots, \text{n}$$

$$V(P,f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \dots + |f(x_n) - f(x_{n-1})|$$

$$V(P,f) = |x_1 - x_0| + |x_2 - x_1| + \dots + |x_n - x_{n-1}|$$

$$V(P,f) = x_1 - x_0 + x_2 - x_1 + \dots + x_n - x_{n-1}$$

$$V(P,f) = x_n - x_0$$

$$V(P,f) = 1 - 0$$

$$V(P,f) = 1$$

Thus , $V(P,f) = \{1\}$, ∀ P ∈ P[a, b]

Hence , Total variation $\sup\{V(P,f) : \forall \text{ P} \in \text{P[a, b]}\} = 1$ i.e. $V_0^1(f) = 1$

3.0.5 Example – not a function of bounded variation

Let $f: [a,b] \to \mathbb{R}$ be a function and

$$f(x) = \begin{cases} 1\,, if\ x \in [0,1] \cap \mathbb{Q} \\ 0, if\ x \in [0,1] \cap \mathbb{Q}^c \end{cases} \qquad (1)$$

Now , first we find variation of the function . Let us take a partition $P = \{x_0, x_1, x_2, \dots, x_{2n}\}$

such that $f(x_{2k}) \in \mathbb{Q}$ and $f(x_{2k-1}) \in \mathbb{Q}^c$

now ,

$$V(P,f) = \textstyle\sum_{j=1}^{2n} |f(x_j) - f(x_{j-1})|$$

$$\Rightarrow V(P,f) = |f(x_2) - f(x_1)| + |f(x_4) - f(x_3)| + \dots + |f(x_{2n}) - f(x_{2n-1})|$$

$$\Rightarrow V(P,f) = |1 - 0| + |1 - 0| + \dots + |1 - 0| \quad (2n - times)$$

$\Longrightarrow V(P,f) = 2n$

$\Longrightarrow V(P,f) \longrightarrow \infty \ \ as \ \ n \longrightarrow \infty$

$\Longrightarrow \sup\{V(P,f): \forall\, P \ \in P(V,f)\} = \infty$

Therefore , this is not a function of bounded variation.

This function is called Dirichlet function , which has infinite variation .

Now , from this example we conclude that a bounded function does not imply that f is a class of *BV- function* .

4 Algebra of bounded variation

If **f** and **g** are function of bounded variation on [a,b] then

- $\boldsymbol{f} + \boldsymbol{g} \in BV[a,b]$ **.**
- $\boldsymbol{f} - \boldsymbol{g} \in BV[a,b]$ **.**
- $\boldsymbol{\alpha f} \in BV[a,b]\ \forall \boldsymbol{\alpha} \in \mathbb{R}$
- $\boldsymbol{f}.\boldsymbol{g} \in BV[a,b]$
- If $1/\boldsymbol{g}$ is bounded on $[a,b]$ then $\boldsymbol{f}/\boldsymbol{g}$ is a class of $BV[a,b]$

5 Important Theorems and Lemmas

5.0.1 Lemma

Let $f:[a,b] \to \mathbb{R}$ be a function and let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition on $[a,b]$ and $P^* = \{y_0, y_1, y_2, \ldots, y_m\}$ be a refinement of P then ,

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le \sum_{j=1}^{m} |f(y_j) - f(y_{j-1})|$$

We first add one extra point z and compute the effect of f by point z in P then

$P_1 = \{x_0, x_1, x_2, \ldots, x_{i-1}, z, x_i, \ldots, x_n\}$.

The interval $[x_{i-1}, x_i]$ divided into two smaller subinterval $[x_{i-1}, z]$ and $[z, x_i]$.

$$V(P,f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \ \ldots + |f(x_i) - f(x_{i-1})| + \ldots + |f(x_n) - f(x_{n-1})|$$

And

$$V(P_1,f) = |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \ \ldots + |f(z) - f(x_{i-1})| + |f(x_i) - f(z)| + \ \ldots + |f(x_n) - f(x_{n-1})|$$

Now, we can write

$|f(x_k) - f(x_{k-1})| = |f(x_k) - f(z) + f(z) - f(x_{k-1})|$

$\Rightarrow |f(x_k) - f(x_{k-1})| \leq |f(x_k) - f(z)| + |f(z) - f(x_{k-1})|$

$\Rightarrow V(P, f) \leq V(P_1, f)$

Similarly, we can continue to adjoining points and examine the effect of it.

$\Rightarrow \forall P^*$(refinement of P) the inequality holds ,

$\Rightarrow V(P, f) \leq V(P^*, f)$

5.0.2 Theorem

Let $f: [a, b] \longrightarrow \mathbb{R}$ be a function of bounded variation on $[a, b]$ and $a < c < b$

i.e. $c \neq a$, $c \neq b$. Then , f is a function of bounded variation on $[a, c]$ and $[c. b]$,

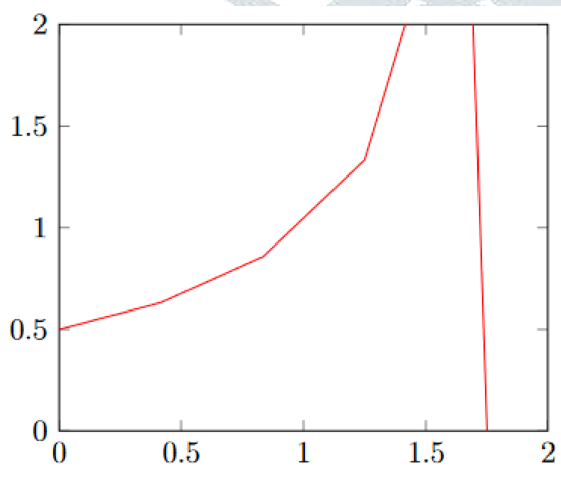
And Total variation $V_a^b(f) = V_a^c(f) + V_c^b(f)$

5.0.3 Example

A function f is of bounded variation on every close subinterval of (a, b) but may not be bounded variation on $[a, b]$.

Let us take a counter example.

$$f(x) = \begin{cases} \frac{1}{2-x}, for\ x \neq 2 \\ 0,\ for\ x = 2 \end{cases} \quad (2)$$



We see that,

The function is increasing on $(0,2)$ and on every close subinterval of interval $(0,2)$ it is a function of bounded variation but as $\rightarrow 2 \Rightarrow f(x) \rightarrow \infty$.

Which means we can take point closer to point 2 and we can increase $f(x)$ as we choose x i.e. $V([0,2], f) = \infty$

$\Rightarrow f(x)$ is not a function of bounded variation on $[0,2]$.

This example is nice to proving that a continuous function on an interval is not implies f is a function of bounded variation .

5.0.4 Monotonicity of $BV-$*function*

Let $f:[a,b] \longrightarrow \mathbb{R}$ be a monotone on $[a,b]$. Then , f is a function of bounded variation on $[a,b]$.

Proof. First we assume that f is monotonic increasing on $[a,b]$.

Let $P = \{y_0, y_1, y_2, \dots, y_n\}$ where $a = y_0 < y_1 < y_2 < \dots < y_n = b$ be a partition on$[a,b]$, then $V(P,f) = |f(y_1) - f(y_0)| + |f(y_2) - f(y_1)| + \dots + |f(y_n) - f(y_{n-1})|$

Since f is monotonic increasing, then

$V(P,f) = f(b) - f(a)$, this satisfies for all$P \in P[a,b]$.

Therefore, the $\sup\{V(P,f): P \in P[a,b]\}$ is $f(b) - f(a) < \infty$ so $f \in BV[a,b]$.

Now, Let f is monotonically decreasing then$V(P,f) = f(a) - f(b)$.

Similarly, by above $\forall P \in P[a,b]$

Therefore, $f \in BV[a,b]$

5.0.5 Example of a discontinuous function which is class of BV −*function*.

$f(x) = [x]$ on interval [0,3]

This example shows that a discontinuous function may be a function of bounded variation but it has only discontinuity of first kind.

5.0.6 Theorem.

. Let $f:[a,b] \longrightarrow \mathbb{R}$ be a function of bounded variation then f can have only countable number of discontinuity. i.e. f have first kind of discontinuity (jump discontinuity).

6 Absolute Continuity.

6.1 Definition.

Let I be an interval in real $\mathbb{R}$. A function $f: I \longrightarrow \mathbb{R}$ is said to absolute continuous on I if

$\forall \epsilon > 0 \ \exists \delta > 0$ Such that

$$\sum_{i=1}^{n} |f(p_i) - f(q_i)| < \epsilon, \text{whenever } [p_i, q_i] \text{ is finite collection of disjoint subinterval}$$

$p_i < q_i \ \exists I$ satisfies $\sum_{i=1}^{n}(p_i - q_i) < \delta$.

6.1.1 Theorem.

An absolute continuous function is uniform continuous but an uniform continuity does not imply absolute continuity.

6.1.2 Lipschitz Function.

Let $f: I \longrightarrow \mathbb{R}$ with I an interval, and $k \in \mathbb{R}$ such that $k > 0$. Then f satisfies a Lipschitz condition with constant k if $|f(b) - f(a)| \leq k|b - a|$ for all $a, b \in I$. Then f is called a Lipschitz function.

6.1.3 Theorem.

If $f : I \longrightarrow \mathbb{R}$ be Lipschitz continuous function then f is an absolute continuous function but an absolute continuous function need not be a Lipschitz continuous function.

First we prove that a Lipschitz continuous is an absolute continuous.

Let $\epsilon > 0$ and choose $\delta = \epsilon/k$. Let us take a set of disjoint close intervals such that $\{[a_i, b_i] : i = 1, 2, 3, \dots n\}$ be a finite partition of interval I.

We have to show that $\sum_{i=1}^{n}(b_i - a_i) < \delta$.

Now, we apply the condition of Lipschitz continuous.

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \sum_{i=1}^{n} k(b_i - a_i) < \delta.$$

We already choose $\delta = \frac{\epsilon}{k}$.

$$\Rightarrow \sum_{i=1}^{n} (b_i - a_i) < \epsilon$$

$$\Rightarrow \sum_{i=1}^{n} |f(b_i) - f(a_i) < \epsilon$$

This is a condition of absolute continuity.

Now, Let us take a counter example to show that absolute continuous functions are not always a Lipschitz continuous.

$f(x) = \sqrt{x}$ is absolute continuous on $[0, 1]$ but not a Lipschitz continuous function.

On applying Lipschitz continuity condition,

$|f(b_i) - f(a_i)| < k|(b_i) - (a_i)|$, where, $[a_i, b_i]; i = 1,2,3, \dots, n$ are disjoint close interval.

$|f(b_i) - f(a_i)| < \left|\sqrt{kb_i} - \sqrt{ka_i}\right|$

$\Rightarrow |f(b_i) - f(a_i)| < \sqrt{k}\left|\sqrt{b_i} - \sqrt{a_i}\right|$.

This does not satisfy Lipschitz continuous condition.

6.1.4 Example of a Uniform continuous function which is not an absolute continuity and not a function of bounded variation

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function such that

$$f(x) = \begin{cases} x\cos\frac{\pi}{x}, when\ x \neq 0 \\ 0, \ when\ x = 0 \end{cases} \qquad (3)$$

We know that this function is uniform continuous on $[0, 1]$.

Define partition $P = \{0, \frac{1}{2n} - \frac{1}{2n-1}, \dots, \frac{1}{2}, 1\}$

Now, $f\left(\frac{1}{2n}\right) = \frac{1}{2n}\cos\frac{2n\pi}{2} = \frac{(-1)^n}{2n}$

And, $f\left(\frac{1}{2n-1}\right) = \frac{1}{2n-1}\cos\frac{(2n-1)\pi}{2} = 0$

$\Rightarrow \left|f\left(\frac{1}{2n}\right) - f\left(\frac{1}{2n-1}\right)\right| = \frac{1}{2n}$

Now, the variation function,

$$V(P,f) = \sum_{j=1}^{n} |f\left(\frac{1}{2j}\right) - f(\frac{1}{2j-1})| = \frac{1}{2}\sum_{j=1}^{n}\frac{1}{j} \longrightarrow \infty$$

$$\Longrightarrow V(P,f) \longrightarrow \infty$$

Thus f is not a function of bounded variation.

6.2 Relation between absolute continuity and bounded variation and also Lipschitz continuous function and bounded variation

6.2.1 Theorem.

If a function f is absolute on the interval I then f is also abounded variation.

6.2.2 Theorem

A Lipschitz continuous function is a bounded variation converse need not be true.

For converse part we can take above example. We have to only prove that a Lipschitz continuous function is always a function of bounded variation.

For this, we apply here Lipschitz continuous function.

$\forall\, y, x \in [a,b] \,\exists\, k \in \mathbb{R}^+$

Such that, $|f(x) - f(y)| \leq k|x - y|$

Let us take partitions, $\forall\, P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b$

Variation of a function,

$$V(P,f) = \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|$$

$$\leq k\sum_{j=1}^{n} |(x_i) - (x_{i-1})|$$

$$\Longrightarrow V(P,f) \leq k(b-a)$$

This shows that a Lipschitz function is always a continuous function.

6.2.3 Theorem related to Lipschitz continuous function and differentiability

.

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function and if $f'(x)$ exists and $f'(x) \leq M$ for some $M > 0$

$\Longrightarrow f$ is Lipschitz continuous.

$\Longrightarrow f \in BV[a, b]$

7 Riemann-Stieltjes Integration.

In this section we define Riemann-Stieltjes integration and discuss some important example. Finally we discuss relationship between Riemann-Stieltjes integral and function of bounded variation.

7.0.1 Definition.

Let $f : [a, b] \longrightarrow \mathbb{R}$ be any real valued bounded function defined on $[a, b]$ and α be a monotonically increasing function on [a, b]. Let P be a partition of $[a, b]$ such that $a = x_0, x_1, x_2, \dots, x_n = b$

$M_i = sup\{f(x) : \forall x \in [x_{i-1}, x_i]\}$

and $m_i = inf\{f(x) : \forall x \in [x_{i-1}, x_i]\}$

Define $U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$

and $L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$

Then $U(P, f, \alpha)$ and $L(P, f, \alpha)$ are called lower Riemann-Stieljes sum and upper Riemann-Stieljes of f with respect to α corresponding to partition P. where,

$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$ such that $\alpha_i \geq 0$, because α is monotonically increasing function.

We define,

$$\overline{\int_a^b} f \, d\alpha = inf_P\{U(P, f, \alpha)\} \qquad (4)$$

And

$$\underline{\int_a^b} f \, d\alpha = sup_P\{L(P, f, \alpha)\} \qquad (5)$$

The equation (4) and (5) are called upper and lower Reimann-Stieljes integral of f with respect to α on $[a, b]$. If

$$\overline{\int_a^b} f \, d\alpha = \underline{\int_a^b} f \, d\alpha \qquad (6)$$

If f satisfies equation (6) then f is Riemann-Stieljes integrable with respect to α on $[a, b]$.

7.0.2 Theorem

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function, and α is monotonically increasing on $[a, b]$. then, prove that the lower Riemann-Stieljes integral of f relative to α can never exceed the upper Riemann-Stieljes integral.

i.e. $\overline{\int_a^b} f \, d\alpha \leq \underline{\int_a^b} f \, d\alpha$

7.0.3 Condition on Riemann-Stieljes integral which is related to function of bounded variation

.Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function on $[a,b]$ and α be monotonic increasing on $[a,b]$. Then, $f \in \mathbb{R}(\alpha) \Leftrightarrow \forall \epsilon > 0 \ \exists$ a partition P on $[a,b]$ such that

$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

Let $f \in \mathbb{R}(\alpha)$ on$[a,b]$ then we have,

$$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha \qquad (7)$$

We know that -

$\underline{\int_a^b} f\,d\alpha = sup_P L(P,f,\alpha)\ \forall P.$

$\exists$ a partition P_1 of $[a,b]$ and a real number $\frac{\epsilon}{2} > 0$ such that

$$L(P,f,\alpha) > \underline{\int_a^b} f\,d\alpha - \frac{\epsilon}{2}$$

$$\Rightarrow -L(P,f,\alpha) < -\underline{\int_a^b} f\,d\alpha + \frac{\epsilon}{2} \qquad (8)$$

Also we know that,

$\overline{\int_a^b} f\ d\alpha = inf_P\{U(P,f,\alpha)\}$

$\exists$ a partition P_2 of $[a,b]$ and a real number $\frac{\epsilon}{2} > 0$ such that

$$\Rightarrow U(P,f,\alpha) < \overline{\int_a^b} f\ d\alpha + \epsilon/2 \qquad (9)$$

Now, we take $P = P_1 \cup P_2$, such that $P_1 \subset P$, $P_2 \subset P$ ..

then, $U(P,f,\alpha) \leq U(P_2,f,\alpha)$.

and $L(P,f,\alpha) \leq L(P_1,f,\alpha)$.

$\Rightarrow -L(P_1,f,\alpha) \geq -L(P,f,\alpha)$

or $-L(P,f,\alpha) \leq -L(P_1,f,\alpha)$

$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) \leq U(P_2,f,\alpha) - L(P_2,f,\alpha) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) \leq \epsilon$

7.0.4 Theorem

If f_1 and f_2 are in $R(\alpha)$ on $[a,b]$, then $f_1 + f_2 \in [a,b]$ and also $f_1 - f_2 \in [a,b]$.

7.0.5 Relation between Riemann-Stieljes and function of bounded variation.

. If f is a function of bounded variation on a close interval $[a, b]$, then we can write f as a difference of two monotonically increasing function by Algebra of bounded variation.

i.e. $f = f_1 + f_2$ where, f_1 and f_2 both increasing function on $[a, b]$.

By the theorem 7.0.4 we can write,

If f_1 and f_2 are two monotonically increasing bounded function then $f_1 - f_2 \in R(\alpha)$ whenever α is continuous and here $f = f_1 - f_2 \in R(\alpha) \implies f \in R(\alpha)$

i.e Function of bounded are like a nice functions that they don't have infinite variation or wiggling. On the other hand, the Riemann-Stieljes is a way to calculate the area under the curve of a function, but it takes into account the variation of the function called Stieljes measure. If we have a function of bounded variation it can always be integrated by the Riemann- Stieljes integral and calculate area under the curve.

8 Rectifiable Curve.

Now, we introduce the concept of Rectifiable curve or arc length. We know that, a straight line is the shortest distance between two points in Euclidean plane. So that the length of any inscribe polygon should not exceed the length of the curve.

Therefore we can say, the length of the curve is work like an upper bound to the length

of all inscribe polygon and the curve should be least upper bound of the length of all possible polygon inscribe polygon.

For most curve, we use in practice will give us useful definition of arc length, but there are some curve for which there are no upper bound.

So, we need to categories curves in two categories- Rectifiable and non-rectifiable.

8.0.1 Curve and Paths.

We know that $[a, b]$ be a compact interval. Let $f : [a, b] \longrightarrow \mathbb{R}^n$ be a vector valued function, which is continuous on compact interval $[a, b]$ in $\mathbb{R}$. Let input value t in $[a, b]$ the function values $f(t)$ trace out a set of points in $\mathbb{R}^n$ called graph of f or curve described by f . We know a curve is compact and connected subset of $\mathbb{R}^n$ since it is continuous image of a compact interval. The function f itself is called a path.

8.0.2 Rectifiable curves in $\mathbb{R}^k$.

Let $f : [a, b] \longrightarrow \mathbb{R}$ and let a partition $P = \{x_0, x_1, x_2, \dots, x_n\}$ of $[a, b]$.

The points $\gamma(x_0), \gamma(x_1), \dots, \gamma(x_n)$ vertices of inscribe polygon and $\Lambda_r(P)$ is the length of the path with vertices $\gamma(x_0), \gamma(x_1), \dots, \gamma(x_n)$.

If the set of number $\Lambda_r(P)$ is bounded for all partition P of $[a, b]$, then the curve γ is said to be rectifiable curve and its arc length is defined as $\Lambda_r([a, b]) = sup_P \Lambda_r(P)$.

In other word the curve γ is said to be rectifiable if $\Lambda_r([a, b]) < \infty$. The length of this polygon is denoted by

$$\Lambda_r(P) = \sum_{k=1}^{m} ||f(r_k) - f(r_{k-1})||$$

If $\Lambda_r(a,b) = \sup\{\Lambda_r(P) : \mathrm{P} \in \mathrm{P[a,b]}\}$ is unbounded then f is called non-rectifiable.

8.0.3 Theorem.

Let $\gamma : [a,b] \longrightarrow \mathbb{R}^k$ be a curve (Rectifiable) and if $c \in (a,b)$ then prove that

$\Lambda_r(a,b) = \Lambda_r(a,c) + \Lambda_r(c,b)$

Let $P = \{a = x_0, x_1, \dots, x_n = b\}$ be a partition of $[a,b]$ adjoining the point $'c'$ to P we get the partition P_1 and P_2 of $[a,c]$ and $[c,b]$.

We get $P_1 = \{a = x_0, x_1, \dots, x_{i-1}, c\}$

and $P_2 = \{c, x_i, x_{i+1}, \dots, x_n = b\}$. such that $\Lambda_r(P) \leq \Lambda_r(P_1) + \Lambda_r(P_2)$

$\Longrightarrow \Lambda_r(a,b) \leq \Lambda_r(a,c) + \Lambda_r(c,b)$

Now, let P_1 and P_2 be arbitrary partitions of $[a,c]$ and $[c,b]$, respectively. Then $P = P_1 \cup P_2$ is a partition of $[a,b]$. then we have,

$$\Lambda_r(P) = \Lambda_r(P_1) + \Lambda_r(P_2)$$

8.0.4 Note.

(a). If γ is injective i.e. one-one then γ is called Jordan arc.

(b). If $\gamma(a) = \gamma(b)$ then γ is said to be closed curve.

It may be noted that we define a curve to be a mapping not a point set and the range of γ.

$$\gamma = \{\gamma(x) \in \mathbb{R}^k;\ \forall x \in [a,b]\}.$$

8.1 Relation between bounded variation and rectifiable curve.

We now consider some theorem and proof to stabilize the relation between bounded variation and arc length.

8.1.1 Theorem.

The length of a curve is finite iff it is of bounded variation.

Proof. Suppose that arc length is finite, for all partition $P = \{x_j : 0 \leq j \leq n\}$.

The arc length is defined as-

$$\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \leq \sum_{j=1}^{n} \sqrt{[(x_j) - (x_{j-1})]^2 + [f(x_j) - f(x_{j-1})]^2}$$

Since arc length is finite, let us suppose $sup\{S\} = M$

Where $S = \sum_{j=1}^{n}\sqrt{((x_j)-(x_{j-1}))^2+(f(x_j)-f(x_{j-1}))^2} : \forall x_j, j = 1,2,3,\dots,n.$

$$\Rightarrow \sum_{j=1}^{n}\sqrt{\left((x_j)-(x_{j-1})\right)^2+\left(f(x_j)-f(x_{j-1})\right)^2} \leq M$$

$$\Rightarrow \sum_{j=1}^{n}\left|f(x_j)-f(x_{j-1})\right| \leq M$$

This is condition of a function to be a class of bounded variation. We conclude from here that the variation of f is less than or equal to M .

This implies f is of bounded variation.

Conversely, Let us suppose that f is of bounded variation and we know that

$\sqrt{a^2+b^2} \leq |a|+|b|.$

$$\Rightarrow \sum_{j=1}^{n}\sqrt{\left((x_j)-(x_{j-1})\right)^2+\left(f(x_j)-f(x_{j-1})\right)^2} \leq \sum_{j=1}^{n}\left|(x_j)-(x_{j-1})\right| + \sum_{j=1}^{n}\left|f(x_j)-f(x_{j-1})\right|$$

$$= b-a+\sum_{j=1}^{n}\left|f(x_j)-f(x_{j-1})\right|$$

$$\leq b-a+V(f,[a,b])$$

Since we suppose that f is of bounded variation this implies that $V\,(f,[a,b])$ is finite and $b-a$ is length of bounded interval then this is also finite and the whole sum is finite,

$$\Rightarrow b-a+V\,(f,[a,b]) < \infty$$

$$\Rightarrow \sum_{j=1}^{n}\sqrt{\left((x_j)-(x_{j-1})\right)^2+\left(f(x_j)-f(x_{j-1})\right)^2} \leq \infty$$

$\Rightarrow$ The length of the curve is finite.

9 Conclusion.

In this paper we enjoy to exploring function of bounded variation. We have also explored some other related topic with bounded variation. However, we did not see them in detail explanation but we have only considered briefly. Here the important and interesting topic is monotonicity of function of bounded variation and that we can write a function of bounded variation in two monotonic increasing functions. We showed that the absolutely continuous function is of function of bounded variation and also considered converse need not be true. We showed by giving example that a uniform continuous function may not be absolutely continuous. Then we showed Riemann-Stieltjes integration is of bounded variation if f can be written as difference of two increasing function. We discussed here that a function of bounded variation have only jump type of discontinuity. We also discussed if f is of function of bounded variation then f is Riemann-Stieltjes integrable over α whenever, α is continuous etc.

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Address:

Dr.Gyanvendra Pratap Singh, Assistant Professor

Department of Mathematics and Statistics

Deen Dayal Upadhyaya Gorakhpur University, Gorakhpur

Shilpi Ojha, M.Sc. Mathematics (IV Sem)

Department of Mathematics and Statistics

Deen Dayal Upadhyaya Gorakhpur University.