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Riemann-Liouville Fractional Derivatives and the Taylor-Riemann Series

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I. ABSTRACT

In this paper we give some background theory on the concept of fractional calculus, in particular the Riemann-Liouville operators. We then investigate the Taylor-Riemann series using Osler's theorem and obtain certain double infinite series expansions of some elementary functions. In the process of this we give a proof of the convergence of an alternative form of Heaviside's series. A Semi-Taylor series is introduced as the special case of the Taylor-Riemann series when, and some of its relations to special functions are obtained via certain generating functions arising in complex fractional calculus.

II. Keywords: Riemann-Liouville operators, Taylor-Riemann series, Heaviside's series, Semi-Taylor series

III. INTRODUCTION

Riemann-Liouville operator. The concept of non-integral order of integration can be traced back the to the genesis of differential calculus itself: the philosopher and creator of modern calculus G.W. Leibniz made some re- marks on the meaning and possibility of fractional derivative of order in the late 17th century. However a rigorous investigation was first carried out by Liouville in a series of papers from 1832-1837, where he defined the first outcast of an operator of fractional integration. Later investigations and further developments by among others Riemann led to the construction of the integral-based Riemann-Liouville fractional integral opera- tor, which has been a valuable cornerstone in fractional calculus ever since. Prior to Liouville and Riemann, Euler took the first step in the study of fractional integration when he studied the simple case of fractional integrals of monomials of arbitrary real order in the heuristic fashion of the time; it has been said to have lead him to construct the Gamma function for fractional powers of the factorial. An early attempt by Liouville was later purified by the Swedish mathematician Holmgren [10], who in 1865 made important contributions to the growing study of fractional calculus. But it was Riemann [4] who reconstructed it to fit Abel's integral equation, and thus made it vastly more useful. Today there exist many different forms of fractional integral operators, ranging from divided-difference types to infinite-sum types, but the Riemann-Liouville Operator is still the most frequently used when fractional integration is performed.

Riemann's modified form of Liouville's fractional integral operator is a direct generalization of Cauchy's formula for an-fold integral,

$$\int_a^x dx_1 \int_a^{x_n} dx_{2\dots} \int_a^{x_{n-1}} f(x_n)\, dx_n = \frac{1}{(n-1)!}\int_a^x \frac{f(t)}{(x-t)^{1-n}}\, dt, \tag{1}$$

and since $(n-1)! = \Gamma(n)$,Riemann realized that the RHS of (1) might have meaning even when takes non-integer values. Thus perhaps it was natural to define fractional integration as follows.

Definition 1. *If* $f(x) \in C([a,b])$ *and* $a < x < b$ *then*

$$I_{a+}^{\alpha} f\,(x) := \frac{1}{\Gamma\,(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \tag{2}$$

Where $\alpha \in]-\infty,\infty[$, is called the Reimann – Liouville fractional integral of order α. In the same fashion for $\alpha \in [0,1]$ we let.

$$D_{a+}^{\alpha} f(x) := \frac{1}{\Gamma\,(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^{\alpha}}\, dt, \tag{3}$$

which is called the Riemann-Liouville fractional derivative of order α .

(It follows from our discussion below Definition 3 that if 0< α < 1 then $D_{a+}^{\alpha} f(x)$ exists for all $f \in C^1([a,b\,])$ and all $x \in [a,b])$

These operators are called the Riemann-Liouville fractional integral operators, or simply R-L operators. The special case of the fractional derivative when $\alpha = \frac{1}{2}$ is called the semi-derivative. The connection between the Riemann-Liouville fractional integral and derivative can, as Riemann realized, be traced back to the solvability of Abel's integral equation for any $\alpha \in [0,1]$

$$f(x) = \frac{1}{\Gamma\,(\alpha)}\int_a^x \frac{\emptyset(t)}{(x-t)^{1-\alpha}}\, dt, x > 0. \tag{4}$$

Formally equation (4) can be solved by changing to and to respectively, further by multiplying both sides of the equation by $(x-t)^{-\alpha}$and integrating we get.

$$\int_a^x \frac{dt}{(x-1)^{\alpha}}\int_a^t \frac{\emptyset(s)ds}{(t-s)^{1-\alpha}} = \Gamma(\alpha)\int_a^x \frac{f(t)dt}{(x-t)^{\alpha}} \tag{5}$$

Interchanging the order of integration in the left hand side by Fubini's theorem we obtain

$$\int_a^x \emptyset(s)ds \int_s^x \frac{dt}{(x-t)^{\alpha}(\,t-s)^{1-\alpha}} = \Gamma(\alpha)\int_a^x \frac{f(t)dt}{(x-t)^{\alpha}} \tag{6}$$

The inner integral is easily evaluated after the change of variable t = s $+\tau(x-s)$ and use of the formulae of the Beta-function :

$$\int_a^x (x-t)^{-\alpha}(t-s)^{\alpha-1}dt = \int_0^1 \tau^{\alpha-1}(1-\tau)^{\alpha}\, d\tau = B(\alpha, 1-\alpha)$$

(7)

Therefore we get

$$\int_a^x \emptyset(s)ds = \frac{1}{\Gamma(1-a)}\int_a^x \frac{f(t)dt}{(x-t)^a} \tag{8}$$

Hence after differentiation we have

$$\emptyset(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)dt}{(x-t)^\alpha} \tag{9}$$

Thus if (4) has a solution it is necessarily given by (9) for any $\alpha \in [0,1]$ One observes that (4) is in a sense the α-order integral and the inversion (9) is the α-order derivative.

A very useful fact about the R-L operators is that they satisfy the following important *semi-group property* of fractional integrals.

Theorem 2. For any $f \in C([a,b])$ the Riemann-Liouville fractional integral satisfies.

$$I_{a+}^{\alpha} I_{a+}^{\beta} f(x) = I_{a+}^{\alpha+\beta} f(x) \tag{10}$$

For $\alpha > 0, \beta > 0$.

Proof. The Proof is rather direct, we have by definition

$$I_{a+}^{\alpha} I_{a+}^{\beta} f(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x \frac{dt}{(x-t)^{1-\alpha}}\int_a^t \frac{f(u)}{(t-u)^{1-\beta}}\,du, \tag{11}$$

and since $f(x) \in C([a,b])$ we can by Fubini's theorem interchange order of integration and by setting $t = u + x(x-u)$ we obtain.

$$I_{a+}^{\alpha} I_{a+}^{\beta} f(x) = \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x \frac{f(u)}{(x-u)^{1-\alpha-\beta}}du = I_{a+}^{\alpha+\beta} f(x) \tag{12}$$

The Riemann-Liouville fractional operators may in many cases be extended to hold for a larger set of, and a rather technical detail is that we denote$\alpha = [\alpha] + \{\alpha\}$, where $[\alpha]$ denotes the remainder the remainder. This notation is used for convenience, observe the following definition.

Definition 3. If $\alpha > 0$ is not an integer then we define.

$$D_{a+}^{\alpha} f = \frac{d^{[\alpha]}}{dx^{[\alpha]}} D_{a+}^{\{\alpha\}} f = \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} I_{a+}^{1-\{\alpha\}} f, \tag{13}$$

thus

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x \frac{f(t)dt}{(x-1)^{\alpha-n+1}} \tag{14}$$

for any $f \in C^{[\alpha]+1}([a,b])$ if $n = [\alpha]+1$. If on the other hand $\alpha < 0$ then the notation.

$$D_{a+}^{\alpha} f = I_{a+}^{-\alpha} f, \tag{15}$$

$May\ be\ used\ as\ definition$

Clearly if $\alpha < 0$ then the fractional derivative $D_{a+}^{\alpha} f(x)$ exists for all $f \in C([a,b])$ and all $x \in [a,b]$. We also remark that for $\alpha > 0$, the fractional derivative $D_{a+}^{\alpha} f(x)$ certainly exists for all $f \in C^{[\alpha]+1}([a,b])$ and all $x \in [a,b]$ (but not necessarily for x = a).

To see this, write $n = (\alpha) + 1$ and apply Taylor's formula with remainder:

$$f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k + \frac{1}{(n-1)!}\int_a^t \frac{f^{(n)}(s)}{(t-s)^{n-1}} ds \qquad \forall t \in [a,b].$$

Substituting this into the definition of $D_{a+}^{\alpha} f(x)$ and simplifying the integrals we obtain.

$$D_{a+}^{\alpha} f(x) = \frac{d^n}{dx^n}\left(\sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+2-\{\alpha\})} . (x-a)^{k+1-\{\alpha\}} + \frac{1}{\Gamma(n+1-\{\alpha\})}\int_a^x f^{(n)}(s). (x-s)^{n-\{\alpha\}} ds\right) \tag{16}$$

Clearly, this $n-$fold derivative can be carried out for all $x \in [a,b]$; in particular the integral is unproblematic since $f^{(n)} \in C([a,b])$ and since the exponent $n - \{\alpha\}$ is larger than $n-1$, so that $\frac{d^k}{dx^k}(x-s)^{n-\{\alpha\}}$ is integrable for all $k = 0,1,\dots,n$. This proves our claim.

For convenience in the later theorems we define the following useful space.

Definition 4. For $\alpha > 0$ let $I_{a+}^{\alpha}([a,b])$ denote the space of functions which can be represented by an R-L integral of order α of some C ([a,b]) – function.

Theorem 5. Let $f \in C([a,b]) and\ a > 0$. In order that $f(x) \in I_{a+}^{\alpha}([a,b])$it is necessary and sufficient that.

$$I_{a+}^{n-\alpha} f \in C^m([a,b]), \tag{17}$$

Where $n = [a] + 1, and\ that$

$$\left(\frac{d^k}{dx^k} I_{a+}^{n-\alpha} f(x)\right)_{|x=a} = 0, k = 0,1,2, \dots\dots, n-1. \tag{18}$$

Proof. First assume $f(x) \in I_{a+}^{\alpha}([a,b])$; then $f(x) = I_{a+}^{\alpha} g(x)$ for some $g \in C([a,b])$. Hence by the semi-group property (Theorem 2) we have

$$I_{a+}^{n-\alpha} f(x) = I_{a+}^{n-\alpha} I_{a+}^{\alpha} g(x) = I_{a+}^{n} g(x) = \frac{1}{(n=1)!}\int_a^x \frac{g(t)}{(x-t)^{1-n}} dt = \int_a^x dx_1 \int_a^{x_1} dx_2 \dots\dots \int_a^{x_{n-1}} g(x_n) dx_n$$

(cf.(1)). This implies that (17) holds, and by repeated differentiation we also see that (18) holds.

Conversely, assume that $f \in C([a,b])$ satisfies (17) and (18). Then by Taylor's formula applied to the function $I_{a+}^{n-\alpha} f$, we have

$$I_{a+}^{n-\alpha} f(t) = \int_a^t \frac{d^n}{ds^n} I_{a+}^{n-\alpha} f(s). \frac{(t-s)^{n-1}}{(n-1)!} ds \ \forall t \in [a,b]$$

Let us write $\varphi(t) = \frac{d^n}{dt^n} I_{a+}^{n-\alpha} f(t)$ then note that $\varphi \in C([a,b])$ by (17). Now by definition 1 and the semi-group property (Theorem 2) the above relation implies

$$I_{a+}^{n-a} f(t) = I_{a+}^{n} \varphi(t) = I_{a+}^{n-a} I_{a+}^{\alpha} \varphi(t),$$

and thus

$$I_{a+}^{n-a} (f - I_{a+}^{\alpha}\varphi) \equiv 0.$$

By the general fact about uniqueness of any solution to Abel's integral equation, and note that we have $n - \alpha > 0$), this implies $f \equiv I_{a+}^{\alpha}\varphi$, and thus $f \in I_{a+}^{\alpha}([a,b])$.

Theorem 6. if $\alpha > 0$ then the equality

$$D_{a+}^{\alpha} I_{a+}^{\alpha} f = f(x) \quad (19)$$

holds for any $f \in C([a,b])$. *Now let* $f \in C^{[a]+1}([a,b])$ *then for the equality*

$$I_{a+}^{\alpha} D_{a+}^{\alpha} f(x) = f(x) \quad (20)$$

to hold we need to assume that f *satisfies the condition in the Theorem 5; otherwise*

$$I_{a+}^{\alpha} D_{a+}^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{a-k-1}}{\Gamma(\alpha-k)} \frac{d^{n-k-1}}{dx^{n-k-1}} (I_{a+}^{n-\alpha} f(x)) \quad (21)$$

holds

proof. By definition we have

$$D_{a+}^{\alpha} f\, I_{a+}^{\alpha} = \frac{1}{\Gamma(\alpha)\Gamma(n-a)} \frac{d^n}{dx^n} \int_a^x \frac{f(s)}{(x-s)^{n-1}} ds. \quad (22)$$

Since the integrals are absolutely convergent we deploy Fubini's theorem and interchange the order of integration and after evaluating the inner integral we obtain

$$D_{a+}^{\alpha} I_{a+}^{\alpha} f = \frac{1}{\Gamma(n)} \frac{d^n}{dx^n} \int_a^x \frac{f(s)}{(x-s)^{n-1}} ds \quad (23)$$

Then (19) follows from (23) by Cauchy's formula (1). Since f in (20) satisfies the conditions in Theorem 5 and $f \in C^{[a]+1}([a,b])$ it follows immediately by (18) that (19) will hold (because the residue term of integration will vanish.) If on the other hand any function $f \in C^{[a]+1}([a,b])$ does not satisfy the condition (18) given in Theorem 5 the residue terms outside the integral will not disappear like in (19), but as integration is deployed (21) is obtain by induction.

Perhaps the second part of theorem 6 is somewhat surprising, and this gives rise to the following interesting corollary.

Corollary 7. Let $\alpha > 0, n \in Z^+$ *and* $f(x) \in C^{[a]+n+1}([a,b])$. Then

$$f(x) = \sum^{n-1} \frac{D_{a+}^{\alpha+k} f(x_0)}{\Gamma(\alpha+k+1)} (x - x_0)^{\alpha+k} + R_{n(x),} \quad (24)$$

For all $\alpha \le x_0 < x \le b$, where

$$R_n(x) = I_{a+}^{\alpha+n} D_{a+}^{\alpha+n} f(x) \quad (25)$$

Is the remainder.

One obtains (24) by deploying I_{a+}^{α} and $D_{a+}^{\alpha} f$ in (16) and rearrange some. Heuristically when letting n and m tend to infinity and if f is a suddiciently good function one obtains the Taylor – Riemann expansion which is a fractional generalization of Taylor's theorem we eilll return to this in section 2. The concept of studying the R-L operator for $\alpha \ge 1$ leads us to the following useful theorem.

An interesting property of the R-L operators is that certain non-differentiable functions such as Weierstrass-function and Riemann- function seem to have fractional derivative of all orders [0,1] see [8] and [7] for investigations on non-differentiability and its relation to fractional calculus. This adds to the problem that the relation between the ordinary derivative and the fractional derivative is not entirely obvious, but the following theorem might give a picture on some of their covariance.

3. APPLICATIONS OF THE TAYLOR-RIEMANN SERIES

In this section we give a proof of an alternative version of Heaviside's exponential series. Further a Semi-Taylor series is introduced as the special case when $\alpha = \frac{1}{2}$ in the Taylor-Riemann series. A Semi-Taylor expansion of the exponential function is also obtained and connections to certain special functions are revealed via the generating functions.

3.1. **Taylor-Riemann expansions.** Firstly we intend to give a proof of an alternate form of Heaviside's series by using D_z^α by means of Osler's theorem.

Theorem 8. *e^{bz} has the following convergent Taylor-Riemann expansion on the ring* $|z-a| = a, z \neq 0$:

$$e^{bz} = \sum_{n=-\infty}^{\infty} \frac{e^{ba} b^{\alpha+n}\big(1 - Q(-\alpha - n, ab)\big)}{\Gamma(\alpha+n+1)} (z-a)^{\alpha+n}, \tag{26}$$

fo

Proof. For the case $\alpha < Z$ we deploy the definition of the complex fractional derivative to e^{bz} and obtain

$$D_0^\alpha f(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z \frac{e^{bt} dt}{(z-t)^{1+\alpha}}, \tag{27}$$

substitute $t = z - u$, thus $du = -dt$ and after some rearrangements one obtains

$$D_0^\alpha f(z) = \frac{1}{\Gamma(-\alpha)} e^{xb} (\int_0^\infty e^{-ub} u^{-1-\alpha} du - \int_0^\infty e^{-ub} u^{-1-\alpha} du, \tag{28}$$

By inspection one sees that (28) contains the incomplete complementary gamma-function:

$$Q(\alpha, z) = \frac{1}{\Gamma(\alpha)} \int_z^\infty e^{-t}\, t^{\alpha-1} dt \tag{29}$$

Inserting it into (28) gives:

$$D_0^\alpha e^{bz} = e^{bz} b^\alpha \,(1 - Q(-\alpha, bz)), \tag{30}$$

which holds for any $Re\,(b) > 0 =$ and any $\alpha \in R$ (30) can be found. Further this fractional derivative expression is equivalent to the Cauchy-type fractional derivative used in Osler's theorem for all $\alpha \in R$ except for $\alpha \in Z$. If we in Osler's theorem let $\sigma = 0$, we obtainthe following Taylor-Riemann expansion for e^{bz} __ on the ring $|z-a| = a$, $Re\,(z) > 0, z \neq a$

$$e^{bz} = \sum_{n=-\infty}^{\infty} \frac{e^{ba} b^{\alpha+n}\big(1 - Q(-\alpha - n, ab)\big)}{\Gamma(\alpha+n+1)} (z-a)^{\alpha+n}, \tag{31}$$

which equals (26). Thus we have shown that the theorem; _ holds for $\alpha < 0$, with some more advanced calculations one can also prove that it holds for $\alpha > 0, \alpha \neq 0,1,2,3$ ...as well. This completes the proof.

3.2. **Semi-Taylor series.** In this section we introduce a Semi-Taylor series as a special case of the Taylor-Riemann series, it appears as a "fractional" power expansion where $\alpha = \frac{1}{2}$, further we show some of its expansions and relations to special functions.

Theorem 9. *Any function which is analytic in some open region containing the disk* $|z-a| \leq |a-b|$ *has a unique fractional power series expansion for z on the circle* $|z-a| = |b-a|, z \neq b$._ *The series appears:*

$$f(z) = \frac{D_{z-b}^{-\frac{1}{2}} f(a)}{\sqrt{\pi}} (z-a)^{-\frac{1}{2}} + \sum_{n=0}^{\infty} \frac{2^{n+1} D_{z-b}^{\frac{1}{2}+n} f(a)}{(2n+1)!!\sqrt{\pi}} (z-a)^{\frac{1}{2}+n} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(2n-3)!!\, D_{z-b}^{\frac{1}{2}-n} f(a)}{2^{n-1}\sqrt{\pi}} (z-a)^{\frac{1}{2}-n},$$

we

Proof. Any analytic function in some open region containing the disk $|z-a| \leq |a-b|$ has a convergent Taylor-Riemann expansion on the circle $|z-a| = |b-a|, z \neq b$.by Osler's theorem (Just let $\sigma = 0$).

Further Lemma proves that the expansion is unique, thus:

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{D_{z-b}^{\alpha+n} f(a)}{\Gamma(\alpha+n+1)} (z-a)^{\alpha+n}, \tag{32}$$

is poi). If we let $\alpha = \frac{1}{2}$ then we can convert the Gamma-function into two cases where it becomes simple relations containing the double-factorial. This leads us to the following reformulation of (32):

$$f(z) = \frac{D_{z-b}^{-\frac{1}{2}} f(a)}{\sqrt{\pi}} (z-a)^{-\frac{1}{2}} + \sum_{n=0}^{\infty} \frac{2^{n+1} D_{z-b}^{\frac{1}{2}+1} f(a)}{(2n+1)!!\sqrt{\pi}} (z-a)^{\frac{1}{2}+n} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(2n-3)!!\, D_{z-b}^{\frac{1}{2}-n} f(a)}{2^{n-1}\sqrt{\pi}} (z-a)^{\frac{1}{2}-n}, \tag{33}$$

which equals (32).

If we as an example deploy our series expansion of the exponential series (17), ie expanding e^{bz} __ in Semi-Taylor, we get the following:

$$e^{bz} = \frac{e^{ba} b^{-\frac{1}{2}} \left(1 - Q\left(\frac{1}{2}, ab\right)\right)}{\sqrt{\pi}} (z-a)^{-\frac{1}{2}} + \sum_{n=0}^{\infty} \frac{2^{n+1} e^{ba} b^{\frac{1}{2}+n} \left(1 - Q\left(-\frac{1}{2} - n, ba\right)\right)}{(2n+1)!!\sqrt{\pi}} (z-a)^{\frac{1}{2}+n} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(2n-3)!!\, e^{ba} b^{\frac{1}{2}-n} \left(1 - Q\left(-\frac{1}{2} - n, ba\right)\right)}{2^{n-1}\sqrt{\pi}} (z-a)^{\frac{1}{2}-n}, \tag{34}$$

The Semi-Taylor series has direct connections to special functions via the generating functions, we give an example below.

In connection with the generating functions given in the Semi-Taylor expansion of $\frac{\cos(\sqrt{z})}{\sqrt{z}}$ by using Bessel function is perhaps surprising

$$\frac{\cos(\sqrt{z})}{\sqrt{z}} = \sum_{m=-\infty}^{\infty} (-1)^m a^{-\frac{m}{2}} J_{|m|}(\sqrt{a}) \big((2|m|-1)!!\big)^{-sgn(m)} (z-a)^{m-\frac{1}{2}} \tag{35}$$

(where we agree that $\big((2|m|-1)!!\big)^{-sgn(m)}$=1 when $m = 0$). This formula is true for all $a > 0$ and all z on the circle $|z - a| = a, z \neq 0$.The relation (35) is obtained by using

$$J_{-\alpha-1}(z)(2z)^{-\alpha-1}\sqrt{\pi} = D_z^{\frac{1}{2}+\alpha} \frac{\cos(\sqrt{z})}{\sqrt{z}} \tag{36}$$

and after deploying and letting the sign function simplify the sum it will become (35).

The Semi-Taylor series may be used to construct otherrelated expansions by using special functions that can be found .

V. CONCLUSIONS

This paper introduced the concept of Riemann-Liouville Fractional Derivatives and the Taylor-Riemann Series, the branch of Mathematics which explores fractional integrals and derivatives. We first gave some basic techniques and functions, such as the Gamma function, the Beta function and the fractional derivative. which were necessary to understand the rest of this paper. Thereafter we proved the construction of the Riemann-Liouville Fractional Derivatives and the Taylor-Riemann Series. Therefore we used the fractional derivative and Cauchy formula for repeated integration respectively. Next we studied Fractional Linear Differential Equations. I also think that the formulas are pretty awkward. It would be a lot harder to compute just a simple integer order derivative or integral. Though it is a very interesting subject and definitely worth researching, I believe it should be left as an 'exotic' branch of Mathematics.

REFERENCES

[1] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and Derivatives: theory and applications, Gordon and Breach, Amsterdam, 1993.

[2] J. L. Lavoie, T. J. Osler, R. Tremblay, Frational Derivatives and Special Functions, SIAM Review, V.18, Issue 2, pp.240-268

[3] E.V. Shchepin, The Uppsala lectures on calculus, Lecture notes, Uppsala 2001.

[4] B.Riemann, Versuch einer allgemeinen Auffassung der Integration und Differentiation, Gesammelte Werke, 1876, pp.62.

[5] T.J. Osler, Leibniz rule, the chain rule and Taylor's theorem for fractional derivatives, Doctoral thesis, New York University, New York, 1970.

[6] G.H. Hardy, Riemann's form of Taylor's series, J. London Math. Soc.,20(1945), pp.48-57. [7] B. A. Faycal, About Non-differentiable Functions, J.Math.Anal.Appl, 263, pp.721-737(2001)

[8] B. Ross, S. Samko and E. Russel Love, Functions that have no first order derivative might have derivatives of all orders less than one, Real. Anal. Exchange 20, No. 2.

[9] G.H. Hardy, Divergent Series, Clarendon press, Oxford, England 1949.

[10] H. Holmgren, Om differentialkalkylen med indecies af hvad natur som helst, Kongl. Svenska Vetenskaps-Akad. Handl.Stockholm, 1865-1866,Bd 5, No.11, 83 pages(in Swedish). [11] E.T. Whittaker, G.N. Watson, A Course of modern analysis, 4:th edition, Ibid, 1962.

[12] J.J. Truiljo, M.Rivero, B.Bonilla, On a Riemann-Liouvill Generalize Taylor's Formula, J.Math.Anal. 231, pp. 255-265 (1999).

[13] Y. Watanabe Notes on the Generalized Derivative of Riemann-Liouville and its Application to Leibnitz's Formula, II, Thoku Math. J., 34, pp.28-41

[14] J.E.Marsden, M.J. Hoffman, Basic Complex Analysis, W.H. Freeman and Company, third edition, 1999.

[15] F.W.J.Olver, The asymptotic expansion of Bessel functions of large order, Philos. Trans. Roy. Soc. London Ser. A 247 (1954), pp.328-368.