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Simple Result on Fixed Point Theorem in Dislocated Quasi -Metric Space

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Abstract

In this paper, Simple Result on fixed point theorem in Complete Dislocated Quasi Metric Space for a pair of continuous mapping using rational inequality has been proved.

Keywords: - dq-metric space, Cauchy Sequence, Complete dq-metric space, Fixed-point.

1. Introduction: -

Let X be a non-empty set and $d: X \times X \rightarrow [0,\infty)$ be a function called dislocated function satisfying following conditions-

1. $d(x,y) = d(y,x) = 0 \Rightarrow x = y$
2. $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x,y,z \in X$

Then d is called Dislocated Quasi Metric on X,if d Satisfies $d(x,y) = d(y,x)$ then it is called Dislocated Metric. [1]

The notion of dislocated topologies is useful in the context of logic programming.

Hitzler and Seda [2] have established a fixed-point theorem in complete dislocated spaces stated below which generalizes well known Banach contraction principle.

Let (X,d) be a Complete d-metric space and let $f: X \rightarrow X$ be a contraction function. Then f has unique fixed-point.

The purpose of this note is to prove Simple Result on Fixed point theorem for a pair of continuous contraction mapping in dislocated quasi metric space using rational inequality.

2. Preliminaries: -

Definition 2.1 [1]: - A Sequence $\{x_n\}$ in dislocated quasi metric space {dq-metric space} (X, d) is called Cauchy sequence if given $\epsilon > 0$, $\exists$ n_0N such that $\forall$ $m, n \geq n_0$

$$d(x_m, x_n) < \epsilon \quad \text{or} \quad d(x_m, x_n) < \epsilon$$

i.e.

$$\min\{d(x_m, x_n), d(x_n, x_m)\} < \epsilon$$

Definition 2.2 [1]: - A Sequence $\{x_n\}$ in dislocated quasi metric converges to x if

$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$ and x is called dq-limit of $\{x_n\}$.

Lemma 2.3 [1] dq-limit in dq-metric space is unique.

Definition 2.4 [1] A dq-metric space (X, d) is called complete if every dq-Cauchy sequence converges in dq-metric space.

Definition 2.5 [1]:- Let (X, d) be a dq-metric space. A map $f: X \to X$ is called contraction if $\exists$ $0 < \lambda < 1$ such that

$$d(Tx, Ty) \leq \lambda d(x, y) \quad \forall x, y \in X$$

3. Main Result: -

Theorem 3.1: - Let (X, d) be a Complete dq-metric space and $S, T: X \to X$ be continuous mapping satisfying.

$$d(Sx, Ty) \leq \alpha \frac{d(x, Sx)[1 + d(x, Sx)]}{1 + d(x, y)} + \beta \frac{d(x, Ty) + d(Ty, Sx)}{1 + d(x, y).\, d(y, Sx)}$$

Where $\forall x, y \in X$ and $0 < \alpha, 0 < \beta$ and $(\alpha + \beta) < 1$. Then S and T have a unique common fixed point.

Proof: - Let $x_0 \in X$ be any arbitrary in X.Define a sequence $\{x_n\}$ in X. Such that

$$S(x_0) = x_1 \quad , \quad T(x_1) = x_2 \quad in\ general$$

$$S(x_{2n}) = x_{2n+1} \quad , \quad T(x_{2n+1}) = x_{2n+2}\,.$$

Consider $d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$

$$\leq \alpha \frac{d(x_{2n}, Sx_{2n}).[1 + d(x_{2n}, Sx_{2n})]}{1 + d(x_{2n}, x_{2n+1})} + \beta \frac{d(x_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, Sx_{2n})}{1 + d(x_{2n}, x_{2n+1}).\, d(x_{2n+1}, Sx_{2n})}$$

$$\leq \alpha \frac{d(x_{2n}, x_{2n+1}).[1 + d(x_{2n}, x_{2n+1})]}{1 + d(x_{2n}, x_{2n+1})} + \beta \frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+2}, x_{2n+1})}{1 + d(x_{2n}, x_{2n+1}).\, d(x_{2n+1}, x_{2n+1})}$$

$$\leq \alpha \frac{d(x_{2n}, x_{2n+1})}{1} + \beta \frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+2}, x_{2n+1})}{1}$$

$$\leq \alpha\, d(x_{2n}, x_{2n+1}) + \beta\, d(x_{2n}, x_{2n+1})$$

$$\leq (\alpha + \beta).\, d(x_{2n}, x_{2n+1})$$

$$d(x_{2n+1}, x_{2n+2}) \leq (\alpha + \beta).\, d(x_{2n}, x_{2n+1})$$

Let $\mathrm{K} = (\alpha + \beta)$ with $0 \leq \mathrm{K} \leq 1$ and $\alpha + \beta < 1$

$$d(x_{2n+1}, x_{2n+2}) \leq \mathrm{K}\, d(x_{2n}, x_{2n+1})$$

Similarly,

$$d(x_{2n}, x_{2n+1}) \leq \mathrm{K}\, d(x_{2n-1}, x_{2n})$$

Continuing like this $d(x_{2n+1}, x_{2n+2}) \leq K^n \, d(x_0, x_1) \to 0 \ as \ 2n \to \infty$

$\because \ \alpha + \beta \leq 1$, Thus $\{x_n\}$ is a Cauchy sequence in a complete dq-metric space X.

So, there is a point $u \in X$ such that $x_n \to u$ Therefore sub sequence $(Sx_{2n}) \to u$ and $(Tx_{2n+1}) \to u$ since S and T are continuous functions so we have $Su = u, Tu = u$

Case-1: - Let u be the common fixed point of S and T then by the condition of the theorem.

$$d(u,u) = d(Su,Tu)$$

$$\leq \alpha \frac{d(u,Su)[1 + d(u,Su)]}{1 + d(u,u)} + \beta \frac{d(u,Tu) + d(Tu,Su)}{1 + d(u,u).d(u,Su)}$$

$$\leq \alpha \frac{d(u,u)[1 + d(u,u)]}{1 + d(u,u)} + \beta \frac{d(u,Su)}{1 + d(u,u).d(u,u)}$$

$$\leq \alpha \, d(u,u) + \beta d(u,u)$$

$$\leq (\alpha + \beta).d(u,u)$$

$$d(u,u).(1 - \alpha - \beta) \leq 0$$

$$\because 1 - \alpha - \beta \neq 0$$

Which gives $d(u,u) = 0$ So $u = u$

Case 2: - Let u, v be fixed points of S and T then $Su = u$, $Tv = v$

$$d(u,v) = d(Su,Tv)$$

$$\leq \alpha \frac{d(u,Su)[1 + d(u,Su)]}{1 + d(u,v)} + \beta \frac{d(v,Tv) + d(Tv,Su)}{1 + d(u,v).d(v,Su)}$$

$$\leq \alpha \frac{d(u,Su)[1 + d(u,u)]}{1 + d(u,v)} + \beta \frac{d(v,Su)}{1 + d(u,v).d(v,Su)}$$

$$d(u,v) \leq \alpha \frac{d(u,Su)[1 + d(u,u)]}{1 + d(u,v)} + \beta \frac{d(v,Su)}{1 + d(u,v).d(v,u)}$$

Replacing $v \ by \ u$ we get

$$d(u,u) \leq \alpha \frac{d(u,Su)[1 + d(u,u)]}{1 + d(u,u)} + \beta \frac{d(u,Su)}{1 + d(u,u).d(u,u)}$$

$$d(u,u) \leq (\alpha+\beta)d(u,u)$$

$$d(u,u).(1-\alpha-\beta) \leq 0$$

$$d(u,u) = 0 \qquad [\because 1-\alpha-\beta \neq 0]$$

Similarly, $d(v,v) = 0$

So $d(u,v) = 0$

Hence $u = v$, this proves the uniqueness.

Theorem 3.2:- Let (X,d) be a Complete dq-metric space and $S,T: X \to X$ be continuous mapping satisfying

$$d(Sx,Ty) \leq \alpha \frac{d(x,Sx)[1+d(x,Sx)]}{1+d(x,y)} + \beta \frac{d(x,Ty)+d(Ty,Sx)}{1+d(x,y).d(y,Sx)} + \gamma \frac{d(x,Sx)^2[\ 1+d(y,Sx)]}{d(x,Ty)+d(Ty,Sx)} + \delta \frac{d(x,Sx)}{1+d(y,Sx)}$$

Where $\forall x,y \in X$ and $0<\alpha, 0<\beta, 0<\gamma, 0<\delta$ and $(\alpha+\beta+\gamma+\delta)<1$. Then S and T have a unique common fixed point.

Proof: - Let $x_0 \in X$ be any arbitrary in X.Define a sequence $\{x_n\}$ in X. Such that

$$S(x_0) = x_1 \quad , \quad T(x_1) = x_2 \quad in\ general$$

$$S(x_{2n}) = x_{2n+1} \quad , \quad T(x_{2n+1}) = x_{2n+2}\,.$$

Consider $d(x_{2n+1}, x_{2n+2}) = d\left(S_{x_{2n}}, T_{x_{2n+1}}\right)$

$$\leq \alpha \frac{d(x_{2n}, Sx_{2n}).[\,1+d(x_{2n}, Sx_{2n})]}{1+d(x_{2n}, x_{2n+1})} + \beta \frac{d(\,x_{2n}\,, Tx_{2n+1}) + d(Tx_{2n+1}, Sx_{2n})}{1+d(x_{2n}, x_{2n+1}).d(x_{2n+1},, Sx_{2n})} + \gamma \frac{d(x_{2n}, Sx_{2n})^2[\ 1+d(x_{2n+1}, Sx_{2n})]}{d(x_{2n}, Tx_{2n+1}) + d(Tx_{2n+1}, Sx_{2n})} + \delta \frac{d(x_{2n}, Sx_{2n})}{1+d(x_{2n+1}, Sx_{2n})}$$

$$\leq \alpha \frac{d(x_{2n}, x_{2n+1}).[\,1+d(x_{2n}, x_{2n+1})]}{1+d(x_{2n}, x_{2n+1})} + \beta \frac{d(\,x_{2n}\,,\ x_{2n+2}) + d(x_{2n+2}, x_{2n+1})}{1+d(x_{2n}, x_{2n+1}).d(x_{2n+1}, x_{2n+1})}$$

$$+\gamma \frac{d(x_{2n}, x_{2n+1})^2[\ 1 + d(x_{2n+1}, x_{2n+1})]}{d(x_{2n}, x_{2n+1})} + \delta \frac{d(x_{2n}, x_{2n+1})}{1 + d(x_{2n+1}, x_{2n+1})}$$

$$\leq \alpha \frac{d(x_{2n}, x_{2n+1})}{1} + \beta \frac{d(x_{2n}, x_{2n+1})}{1} + \gamma \frac{d(x_{2n}, x_{2n+1})[1 + d(x_{2n+1}, x_{2n+1})]}{1} + \delta \frac{d(x_{2n}, x_{2n+1})}{1}$$

$$\leq \alpha d(x_{2n}, x_{2n+1}) + \beta d(x_{2n}, x_{2n+1}) + \gamma d(x_{2n}, x_{2n+1}) + \delta d(x_{2n}, x_{2n+1})$$

$$d(x_{2n+1}, x_{2n+2}) \leq (\alpha + \beta + \gamma + \delta) d(x_{2n}, x_{2n+1})$$

Let $\lambda = (\alpha + \beta + \gamma + \delta)$ with $0 \leq \lambda \leq 1$ and $\alpha + \beta + \gamma + \delta < 1$

$$d(x_{2n+1}, x_{2n+2}) \leq \lambda\, d(x_{2n}, x_{2n+1})$$

Similarly, $d(x_{2n}, x_{2n+1}) \leq \lambda\, d(x_{2n-1}, x_{2n})$

Continuing like this $d(x_{2n+1}, x_{2n+2}) \leq \lambda^n\, d(x_0, x_1) \to 0\ as\ 2n \to \infty$

$\because$ $(\alpha + \beta + \gamma + \delta) \leq 1$, Thus $\{x_n\}$ is a Cauchy sequence in a complete dq-metric space X.

So, there is a point $u \in$ X such that $x_n \to u$ Therefore sub sequence $(Sx_{2n}) \to u$ and $(Tx_{2n+1}) \to u$ since S and T are continuous functions so we have $Su = u, Tu = u$

Now, the proof of the theorem is similar to theorem3.1

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