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MADHYA OF A MULTISET GROUP AND ITS STRUCTURAL PROPERTIES

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ABSTRACT

Multiset theory provides an effective generalization of classical set theory by allowing repeated occurrences of elements through frequency functions. When multisets are defined over algebraic structures such as groups, rich and meaningful extensions arise. In this paper, a new concept called *Madhya of an Mgroup* is introduced and studied in detail. The notion is motivated by the ideas of abelian groups and abelian *M*groups, where commutativity plays a central role. The Madhya of an *M*group is defined using symmetry conditions on frequency functions, and several illustrative examples are discussed over well-known groups such as the quaternion group, power set groups under symmetric difference, and permutation groups. Important properties of Madhya are investigated, including its relationship with the original *M*group. A series of theorems are proved to establish that Madhya forms a meaningful and structurally significant sub-multiset of an *M*group. These results strengthen the link between multiset theory and group theory and open new directions for further research in multiset-based algebraic structures.

Keywords: *Mgroup, Multiset theory, Frequency functions and Madhya Mgroup etc.*

Introduction:

Multiset theory extends classical set theory by permitting multiple occurrences of elements, each represented by a frequency function. This flexibility makes multisets highly suitable for applications in algebra, computer science, and information systems. When multisets are defined over groups, the resulting structure, known as a multiset group or *M*group, combines algebraic operations with frequency-based behaviour. Frequency plays a crucial role in determining structural properties of an *M*group. In classical group theory, commutativity leads to the important concept of abelian groups, while in multiset theory, abelian *M*groups are defined through symmetry of frequency functions. Motivated by these ideas, this paper introduces a new concept called the *Madhya of an Mgroup*, which captures elements whose frequency behaviour remains symmetric under group operation. The Madhya of an *M*group is constructed by retaining elements that satisfy frequency commutativity and discarding those that violate it. Through various examples and frequency operation tables, the behaviour of Madhya is illustrated in both abelian and non-abelian settings. Several theorems are established to study its fundamental properties, such as inclusion relations, existence of identity, inverse elements and also proved how it satisfies closure property and how to form *M*group. This study provides a deeper understanding of symmetry in *M*groups and enriches the theoretical foundation of multiset algebra.

In this paper we have introduced definitions of a newly concepts i.e, Madhya of an *M*group, and we have discussed some theorems related to this.

We know that if $(U,*)$ be a group such that for any two elements u,v in U, $u*v = v*u$, then the group U is called an abelian group, also we know that the frequency plays an important role for an *M*set. Some researchers have defined an abelian *M*group in such a way that any *M*group Υ taken from a group $(U,*)$ is said to be an abelian *M*group over U if, $Ƒ_{\Upsilon}(u*v) = Ƒ_{\Upsilon}(v*u)$ for all $u,v \in U$.

Definitions of abelian group and abelian *M*group help us to create two important concepts which are given in this paper. Some examples and theorems are discussed here.

Definition 1.1: *M*set [8]:

A multiset or an *M*set Y is a collection of some or all elements over an ordinary set T with their frequencies or repetitions. Frequency of any member $t \in T$ is denoted by $Ƒ_{\mathbf{Y}}(t)$. We denote *Mset* space as $[T]^p$, is the collection of all possible *M*sets whose elements are from T with maximum frequency p.

Example 1.1:

Suppose T= {p, q, r, s} and Y= {‹2, p›, ‹4, r›}.

Frequencies of p, q, r, s are 2, 0, 4, 0 respectably.

So, $Ƒ_{\mathbf{Y}}(p) = 2$, $Ƒ_{\mathbf{Y}}(r) = 4$, $Ƒ_{\mathbf{Y}}(q) = 0$ etc.

Definition 1.2: Sub *M*set [8]:

Suppose Y, Z $\in [T]^p$ then Yis called a sub *M*set of Z, denoted by $Y \subseteq Z$ if, $Ƒ_Z(t) \geq Ƒ_Y(t)$ for all t $\in$ T. In addition, if there exist at least one t $\in$ T such that, $Ƒ_Y(t) < Ƒ_Z(t)$, then we say that Y is a proper sub *M*set of Z which is denoted by $Y \subset Z$.

Definition 1.3: *M*group [7]:

Suppose Y is an *M*set taken from a group (U,⁎), if it satisfies following two conditions:

1. $Ƒ_Y(t * r) \geq MIN\ \{Ƒ_Y(t), Ƒ_Y(r)\}$; for all t, r $\in$ U.

2. $Ƒ_Y(t^{-1}) = Ƒ_Y(t)$ for any t $\in$U.

Then Y is an *M*group over (U,⁎).

Example 1.2:

If we consider set T= $\{1, ɯ, ɯ^2\}$ where ɯ is an imaginary cube root of unity, then with respect to the usual multiplication of it forms a group.

Let Y= $\{<5, 1>, <4, ɯ^2>, <4, ɯ>\}$. From definition, it can be easily shown that Y is an *M*group.

Theorem 1.1 [9]:

If (U, ⁎) is a group with identity element e and ϒ is an *M*set taken from U then, $Ƒ_ϒ(e) \geq Ƒ_ϒ(u)$ for all u $\in$U.

Theorem 1.2 [9]:

Let ϒ is an *M*set taken from a group U, then

$ϒ \in MG[Ʊ] \Leftrightarrow Ƒ_ϒ(u * v^{-1}) \geq \min\ \{Ƒ_ϒ(u), Ƒ_ϒ(v)\}, \forall\ u, v \in U$.

MAIN DEFINITION AND RESULTS:

Definition 2.1: Madhya of an *M*group:

Let (U,⁎) be a group and ϒ be an *M*group over U.

Then Madhya of ϒ is denoted by $\mathcal{M}(ϒ)$is an *M*set such that,

$Ƒ_{\mathcal{M}(ϒ)}(u) = Ƒ_ϒ(u)$, u $\in$ U if $Ƒ_ϒ(u * v) = Ƒ_ϒ(v * u)\ \forall\ v \in U$.

$Ƒ_{\mathcal{M}(ϒ)}(u) = 0$, u $\in$ U if $Ƒ_ϒ(u * v) \neq Ƒ_ϒ(v * u)$ for some v $\in$U.

Example 2.1: Q_8 ={±1, ±i, ±j, ±k} & Υ={<4,1>,<2,-1>,<2,i>,<2,-i>}

.	1 (4)	-1 (2)	i (2)	-i (2)	j (0)	-j (0)	k (0)	-k (0)
1 (4)	1 (4)	-1 (2)	i (2)	-i (2)	j (0)	-j (0)	k (0)	-k (0)
-1(2)	-1(2)	1 (4)	-i (2)	i (2)	-j (0)	j (0)	-k (0)	k (0)
i (2)	i (2)	-i (2)	-1 (2)	1 (4)	k (0)	-k (0)	-j (0)	j (0)
-i (2)	-i (2)	i (2)	1 (4)	-1(2)	-k (0)	k (0)	j (0)	-j (0)
j (0)	j (0)	-j (0)	-k (0)	k (0)	-1(2)	1 (4)	i (2)	-i (2)
-j (0)	-j (0)	j (0)	k (0)	-k (0)	1 (4)	-1 (2)	-i (2)	i (2)
k (0)	k (0)	-k (0)	j (0)	-j (0)	-i (2)	i (2)	-1 (2)	1 (4)
-k (0)	-k (0)	k (0)	-j (0)	j (0)	i (2)	-i (2)	1 (4)	-1 (2)

Clearly Υ is an *M*group and the above table is symmetric with respect to frequency,

here, $\mathcal{M}$(Υ)= Υ = {<4,1>,<2,-1>,<2,i>,<2,-1>}

Example 2.2:

Let A= {1,2}, Take U=P(A)

i.e. Power set of A with symmetric difference.

Therefore, U = {Φ, {1}, {2}, {1,2}}

clearly U is a group with respect to operation symmetric difference.

Let Υ= {<3, Φ>,<2,{1}>,<2,{2}>,<3,{1,2}>}

Frequency operation table is:

Δ	Φ 3	{1} 2	{2} 2	{1,2} 3
Φ 3	Φ 3	{1} 2	{2} 2	{1,2} 3
{1} 2	{1} 2	Φ 3	{1,2} 3	{2} 2
{2} 2	{2} 2	{1,2}3	Φ 3	{1} 2
{1,2} 3	{1,2} 3	{2} 2	{1} 2	Φ 3

Here, $\mathcal{M}$(Υ)=Υ={<3,Φ>,<2,{1>,<2,{2}>,<3, {1,2}>}.

Example 2.3: Permutation group S_3

i.e., S_3= {e, α, β, γ, δ, σ}

Where, $\alpha = (1\ 2\ 3)$, $\beta = (1\ 3\ 2)$,

$\gamma = (2, 3)$, $\delta = (1\ 2)$,

$\sigma = (1\ 3)$.

Let $\Upsilon = \{<4,e>,<3,\alpha>,<3,\beta>,<3,\delta>,<4,\sigma>,<4,\gamma>\}$.

Frequency operation table is:

.	e (4)	α (3)	β (3)	γ (4)	δ (3)	σ (4)
e (4)	e (4)	α (3)	β (3)	γ (4)	δ (3)	σ (4)
α (3)	α (3)	β (3)	e (4)	δ (3)	σ (4)	γ (4)
β (3)	β (3)	e (4)	α (3)	σ (4)	γ (4)	δ (3)
γ (4)	γ (4)	σ (4)	δ (3)	e (4)	β (3)	α (3)
δ (3)	δ (3)	γ (4)	σ (4)	α (3)	e (4)	β (3)
σ (4)	σ (4)	δ (3)	γ (4)	β (3)	α (3)	e (4)

Clearly, Υ is an *M*group over S_3.

Here 1st row and 1st column are symmetric.

Also 5th row and 5th column are symmetric.

Therefore, $\mathcal{M}(\Upsilon) = \{<4,e>,<3,\delta>\}$

Theorem 2.1:

If $(U,*)$ is a group and Υ is an *M*group over U, then $\mathcal{M}(\Upsilon) \subseteq \Upsilon$.

Proof:

Case-1: Let $u \in U$ with $Ƒ_{\Upsilon}(u) = r$ where $r \in \mathbf{N} \cup \{0\}$

and $Ƒ_{\Upsilon}(u*v) = Ƒ_{\Upsilon}(v*u)\ \forall\ v \in U$.

Then, by definition $Ƒ_{\mathcal{M}(\Upsilon)}(u) = Ƒ_{\Upsilon}(u)$.

Case-2: Let $u \in U$, such that $Ƒ_{\Upsilon}(u) = r$ where $r \in$ **N.**

and $Ƒ_{\Upsilon}(u*v) \neq Ƒ_{\Upsilon}(v*u)$ for some $v \in U$,

then, $Ƒ_{\mathcal{M}(\Upsilon)}(u) = 0 < Ƒ_{\Upsilon}(u)$.

Case-3: Let u Є U, such that $Ƒ_{ϒ}(u)= 0$ and $Ƒ_{ϒ}(u*v)\neq Ƒ_{ϒ}(v*u)$ for some v Є U, then $Ƒ_{\mathcal{M}(ϒ)}(u) =0 = Ƒ_{ϒ}(u)$.

Therefore, in any case $Ƒ_{\mathcal{M}(ϒ)}(u)\leq Ƒ_{ϒ}(u)$ for any u Є U. Thus, $\mathcal{M}(ϒ)\subseteq ϒ$.

Theorem 2.2:

Suppose, (U,⁎) is an abelian group, ϒ is an *M*group over U and U=$ϒ^*$ then $\mathcal{M}(ϒ)=ϒ$.

Proof:

since, (U,⁎) is an abelian group.

Then for any fixed u Є U, u⁎v=v⁎u ∀ v Є U.

⟹$Ƒ_{ϒ}(u*v) = Ƒ_{ϒ}(v*u)$ ∀ v Є U.

⟹u Є $\mathcal{M}(ϒ)$.

Now since U=$ϒ^*$so, for all u Є U, $Ƒ_{ϒ}(u)> 0$

This implies, $Ƒ_{\mathcal{M}(ϒ)}(u) = Ƒ_{ϒ}(u) > 0$ for all u Є U.

Therefore $\mathcal{M}(ϒ)=ϒ$.

Theorem 2.3:

If (Ʊ,⁎) is a group and ϒ is a non-empty *M*group taken from U then identity element e $Є^{r}$$\mathcal{M}(ϒ)$ where r Є N.

Proof:

Since ϒ is an *M*group taken from U.

So, $Ƒ_{ϒ}(e) \geq Ƒ_{ϒ}(u)$ ∀ u Є U

Therefore, if ϒ is a non-empty *M*group,

then $Ƒ_{ϒ}(e)=r$, for a specific r ϵ **N.**

Now we know that, e⁎v=v⁎e∀ v Є U.

This implies, $Ƒ_{ϒ}(e*v) = Ƒ_{ϒ}(v*e)$ ∀ v Є U.

As ϒ is non-empty and we know,

$Ƒ_{ϒ}(e) \geq Ƒ_{ϒ}(u)$ ∀ u Є U.

So, e must be element of ϒ with positive frequency.

Therefore, e $Є^{r}$$\mathcal{M}(ϒ)$ where r Є **N.**

Theorem 2.4:

If (U,⁎) is a group and ϒ is a non-empty *M*group taken from U.

Then u $\in \mathcal{M}(\Upsilon)$ with positive frequency.

$\Rightarrow$ $u^{-1} \in \mathcal{M}(\Upsilon)$ with positive frequency greater.

Proof:

If u $\in \mathcal{M}(\Upsilon)$ with positive frequency,

i.e. $\text{Ƒ}_{\mathcal{M}(\Upsilon)}(u)>0$

$\Rightarrow$ $\text{Ƒ}_{\Upsilon}(u*v) = \text{Ƒ}_{\Upsilon}(v*u)\ \forall\ v \in U.$

$\Rightarrow$ $\text{Ƒ}_{\Upsilon}(u*v^{-1}) = \text{Ƒ}_{\Upsilon}(v^{-1}*u)$ [putting v^{-1} in the place of v as $v^{-1} \in U$]

$\Rightarrow$ $\text{Ƒ}_{\Upsilon}((u*v^{-1})^{-1}) = \text{Ƒ}_{\Upsilon}((v^{-1}*u)^{-1})\ \forall\ v^{-1} \in U.$ [from definition of *M*group]

$\Rightarrow$ $\text{Ƒ}_{\Upsilon}((v^{-1})^{-1}*u^{-1}) = \text{Ƒ}_{\Upsilon}(u^{-1}*(v^{-1})^{-1})\ \forall\ v^{-1} \in U.$

$\Rightarrow$ $\text{Ƒ}_{\Upsilon}(v*u^{-1}) = \text{Ƒ}_{\Upsilon}(u^{-1}*v)\ \forall\ v \in U.$

Therefore, $u^{-1} \in \mathcal{M}(\Upsilon)$ and $\text{Ƒ}_{\mathcal{M}(\Upsilon)}(u)= \text{Ƒ}_{\Upsilon}(u)= \text{Ƒ}_{\Upsilon}(u^{-1}) =\text{Ƒ}_{\mathcal{M}(\Upsilon)}(u^{-1})> 0.$

Theorem 2.5:

If (U,$*$) is a group and Υ is a non-empty *M*group over U. Suppose u & v are two elements of $\mathcal{M}(\Upsilon)$ with positive frequency, then u$*$v also elements of $\mathcal{M}(\Upsilon)$ with positive frequency.

Proof:

Since u and v are two elements of $\mathcal{M}(\Upsilon)$ with positive frequency

So $\text{Ƒ}_{\mathcal{M}(\Upsilon)}(u)>0$ & $\text{Ƒ}_{\mathcal{M}(\Upsilon)}(v)>0$

With $\text{Ƒ}_{\Upsilon}(u*z) = \text{Ƒ}_{\Upsilon}(z*u)\ \forall\ z \in U$ [a]

and $\text{Ƒ}_{\Upsilon}(v*z) = \text{Ƒ}_{\Upsilon}(z*v)\ \forall\ z \in U$ [b]

Now $\text{Ƒ}_{\mathcal{M}(\Upsilon)}(u) = \text{Ƒ}_{\Upsilon}(u) >0,$

Also $\text{Ƒ}_{\mathcal{M}(\Upsilon)}(v) = \text{Ƒ}_{\Upsilon}(v) >0$

We know, if u,v $\in \mathcal{M}(\Upsilon)$ with frequency greater than zero

Then, $u^{-1}, v^{-1} \in \mathcal{M}(\Upsilon)$ with frequency greater than zero.

Therefore, $\text{Ƒ}_{\Upsilon}(u^{-1}*z) = \text{Ƒ}_{\Upsilon}(z*u^{-1})\ \forall\ z \in U$ [c]

and $\text{Ƒ}_{\Upsilon}(v^{-1}*z) = \text{Ƒ}_{\Upsilon}(z*v^{-1})\ \forall\ z \in U$ [d]

Now $\text{Ƒ}_{\Upsilon}((u*v)^{-1}*z) = \text{Ƒ}_{\Upsilon}((v^{-1}*u^{-1})*z)$

$= Ƒ_{\Upsilon}(v^{-1}*(u^{-1}*z))$

$= Ƒ_{\Upsilon}((u^{-1}*z)*v^{-1})$ [from (d)]

$= Ƒ_{\Upsilon}(u^{-1}*(z*v^{-1}))$

$= Ƒ_{\Upsilon}((z*v^{-1})*u^{-1})$ [from (c)]

$= Ƒ_{\Upsilon}(z*(v^{-1}*u^{-1}))$

$= Ƒ_{\Upsilon}(z*(u*v)^{-1})$

Therefore, $(u*v)^{-1}$ is member of $\mathcal{M}(\Upsilon)$, that means $u*v$ is also member of $\mathcal{M}(\Upsilon)$.

and $Ƒ_{\mathcal{M}(\Upsilon)}(u*v) = Ƒ_{\Upsilon}(u*v) \geq min\ \{Ƒ_{\Upsilon}(u), Ƒ_{\Upsilon}(v)\} > 0$.

Hence, $u*v$ also elements of $\mathcal{M}(\Upsilon)$ with positive frequency.

Theorem 2.6:

If (Ʊ,∗) is a group and ϒ is a non-empty *M*group over U then

$\mathcal{M}(\Upsilon)$ is also a sub *M*group of ϒ.

Proof:

Already we have proved that**,** $\mathcal{M}(\Upsilon) \subseteq \Upsilon$.

Let u & v are two elements of U such that,

Case-1:$Ƒ_{\mathcal{M}(\Upsilon)}(u) = 0$ &$Ƒ_{\mathcal{M}(\Upsilon)}(v) \neq 0$

Then, obviously $Ƒ_{\mathcal{M}(\Upsilon)}(u*v^{-1}) \geq min\ \{Ƒ_{\mathcal{M}(\Upsilon)}(u), Ƒ_{\mathcal{M}(\Upsilon)}(v)\}$.

Case-2:$Ƒ_{\mathcal{M}(\Upsilon)}(u) \neq 0$ &$Ƒ_{\mathcal{M}(\Upsilon)}(v) = 0$

Then, also $Ƒ_{\mathcal{M}(\Upsilon)}(u*v^{-1}) \geq min\ \{Ƒ_{\mathcal{M}(\Upsilon)}(u), Ƒ_{\mathcal{M}(\Upsilon)}(v)\}$.

Case-3:$Ƒ_{\mathcal{M}(\Upsilon)}(u) = 0$ &$Ƒ_{\mathcal{M}(\Upsilon)}(v) = 0$

Again, we have,$Ƒ_{\mathcal{M}(\Upsilon)}(u*v^{-1}) \geq min\ \{Ƒ_{\mathcal{M}(\Upsilon)}(u), Ƒ_{\mathcal{M}(\Upsilon)}(v)\}$.

Case-4:$Ƒ_{\mathcal{M}(\Upsilon)}(u) \neq 0$ &$Ƒ_{\mathcal{M}(\Upsilon)}(v) \neq 0$

So $Ƒ_{\mathcal{M}(\Upsilon)}(u) = Ƒ_{\Upsilon}(u) > 0$ &$Ƒ_{\mathcal{M}(\Upsilon)}(v) = Ƒ_{\Upsilon}(v) > 0$

Therefore, $Ƒ_{\mathcal{M}(\Upsilon)}(u^{-1}) > 0$ &$Ƒ_{\mathcal{M}(\Upsilon)}(v^{-1}) > 0$

and, by previous Theorem, $Ƒ_{\mathcal{M}(\Upsilon)}(u*v^{-1}) > 0$.

Also $Ƒ_{\mathcal{M}(\Upsilon)}(u * v^{-1}) = Ƒ_{\Upsilon}(u * v^{-1})$

$\geq min\ \{Ƒ_{\Upsilon}(u), Ƒ_{\Upsilon}(v)\}.$

$= min\ \{Ƒ_{\mathcal{M}(\Upsilon)}(u), Ƒ_{\mathcal{M}(\Upsilon)}(v)\}$

Therefore, $Ƒ_{\mathcal{M}(\Upsilon)}(u * v^{-1}) \geq min\ \{Ƒ_{\mathcal{M}(\Upsilon)}(u), Ƒ_{\mathcal{M}(\Upsilon)}(v)\}.$

So, for any case of above $Ƒ_{\mathcal{M}(\Upsilon)}(u * v^{-1}) \geq min\ \{Ƒ_{\mathcal{M}(\Upsilon)}(u), Ƒ_{\mathcal{M}(\Upsilon)}(v)\}$ for all u, v Є U.

Hence, $\mathcal{M}(\Upsilon)$ is a sub *M*group of Υ.

Conclusion

In this paper, the concept of *Madhya of an Mgroup* has been introduced and systematically analyzed. The Madhyaserves as an important substructure within multiset group theory. It has been shown that the Madhya of an *M*group is always a sub-multiset of the original *M*group and in the case of abelian groups, coincides entirely with the *M*group itself. The presence of the identity and inverse element in the Madhya of any non-empty *M*group was established also proved that, it forms an *M*group. These, results confirm that the concept of Madhya is algebraically meaningful and closely aligned with classical group-theoretic notions. The study enhances the interaction between multiset theory and group theory and provides a framework for extending similar ideas to other algebraic structures such as rings, semigroups, and fuzzy multisets.

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