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Fractional Calculus in Mathematical Modelling

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Abstract: Using fractional calculus modelling the memory effect in the model is allowed. With the present data mathematical models can help to forecast future outbreak of diseases. Depending on the disease of interest, different methods and more convenient methods are to be a Studying the dynamics of predator-prey model consisting of spatial interactions among the prey, intermediate predator and the top predator as well as formulating and analysing a new mathematical model for COVID-19 epidemic is the main target. A new model for COVID-19 epidemic with isolated class in fractional order is formulated and analysed. Main objective of ' Disease Modelling using Fractional Differential Equations and Estimation' is to show that FDEs based on concrete examples and experimental data may model more effectively specific problems than ODEs. By using the Caputo- Fabrizio fractional operator the SIRS-SI malaria model is modified for memory effect inclusion . Focusing on reviewing the results of epidemiological modelling including fractional susceptible-Infective-Recovered SIR, SEIR, SEIR model is one objective of this paper. The fixed point theory is used on a paper to examine the existence and uniqueness of solution of the considered fractional SIRS-SI model describing the spreading of malaria. Various methods for approximate solutions of system of non linear differential equations such as Finite decomposition method (FDM), Variational interation method (VIM), Finite element method (FEM), Adomian decomposition method (ADM), Homotopy perturbation method (HPM), Homotopy analysis method (HAM) are applied on another paper. To explore the transmission of chlamydia, various modelling has been presented and mathematical modelling of chlamydia trachomatis in human carrier was introduced. Fractional calculus is used to depict on SEIR model with a system of fractional –order differential equations. The modified hybrid Nelder Mead Simplex search and particle swarm optimization (MH-NMSS-PSO) algorithm to estimate the parameters for fractional DEs and the multi- term – fractional DEs are employed. An optimal control system for the mosaic epidemic in terms of Caputo fractional derivatives is derived. The generalized Euler method (GEM) is found to be powerful, effective and reliable method for investigation of fractional mathematical model. The multi- term fractional SEIAR model provides better fit to the real data than the integer order and single term fractional SEIAR models. A new mathematical model for COVID-19 epidemic with isolated class helps to understand the mechanism of the transmission , characteristics of diseases and to protect population health. A fractional SIRS-SI malaria model transmission along with a no. of cures is studied. It is predicted that the proposed fractional calculus model provides a wide scope for analysing diseases scientifically. The results of epidemiology modelling mainly for the fractional epidemic model is reviewed such as mathematical modelling of dengue epidemic (SIR-SI MODEL), mathematical modelling of HIV/AIDS epidemic (SIJA MODEL).

Key words: Predator, Chlamydia, mathematical model, COVID-19, SIR, special function , asymptotically stable, compartmental model.

INTRODUCTION:

The first mathematician who researched in Fractional Calculus in 1730 was Euler. Then for the theoretical development of FC a number of mathematicians like Lagrange (1772), Laplace (1812), Lacroix (1819), Fourier (1822), Riemann (1847), Laurent (1884) worked in front line and other physicists, engineers used the applications in their scope. A light of FC came after collaborating and published a book devoted to FC in 974 by Oldhem (chemist) and Spnier (a mathematician). The first international conference about FC was organised at University of New Heven, U.S. A, in 1974.

Chlamydia, the silent epidemic in [8] is one of the most generic sexually transmitted epidemics in the United States and European nations. Chlamydia having unique and complex biphasic growth cycle is found to exist in a chlamydial persistence condition.[4] Since the functions like the gamma, beta, and Metteg- Leffler functions are involved in fractional calculus, introducing some special functions is important.

Gamma Function: The Gamma function, denoted by $\Gamma(z)$ is a generalization of the fractional function when z is not an integer.

For $z > 0$, $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$

The Mittag- Leffler Function: It is the generalization of exponential function.

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \ \alpha \in \mathbb{R}^+, \ z \in \mathbb{C},$$

For two parameters,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \ \alpha, \beta \in \mathbb{R}^+, \ z \in \mathbb{C},$$

Considering an example of FDE:

Fractional order initial value problem given by $D^{\frac{1}{2}} x(t) = t, \quad x(0) = x_0.$

$$g(v) = 2\sqrt{t}\Gamma\frac{3}{2}v - v^2\Gamma^2\left(\frac{3}{2}\right).$$

Solution of the corresponding integer- order initial value problem is,

$$x_1(v) = \sqrt{t}\Gamma\left(\frac{3}{2}\right)v^2 - \frac{v^3\,\Gamma^2\frac{3}{2}}{3} + x_0.$$

Solution of the given fractional order initial value problem is

$$x(t) = x_1\left(\frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\right) = \frac{4t^{\frac{3}{2}}}{3\sqrt{\pi}} + x_0.$$

According to WHO, malaria is a major high –risk infectious illness. Many scientists and mathematicians suggested many mathematical models for malaria transmission. A modified mathematical modelling with the assumption that persons of recovered group have the chance to be proned. Recently the HAM is employed by Senthamarai et al to inspect the spreading of malaria illness in an SIRS-SI model. Almost approaches and mathematical models being local nature of integer-order derivatives, fractional derivative approaches are proposed in mathematical modelling of biological and physical systems. To the mathematicians and researchers, FC become the latest and useful tool for its applications in different fields. From time to time, the integrity of FC can be observed in research and modelling of various fields.

Fractional order mathematical model about drug resistance tuberculosis with two line treatment is discussed in this paper by applying Caputo - fractional derivative and this model is solved by using generalized Euler method (GEM). Analysis of mathematical model for giving awareness knowledge in the world is becoming an integral part of this treatment. A new mathematical model for COVID-19 epidemic is formulated and analysed in [9]. A system of fractional order DEs model describes this model including five classes –S(Susceptible class), E(Exposed class), I (Infected class), Q(Isolated class), and R(Recovered class).

A. Analysis on [8] " **Role of fractional derivatives in the mathematical modelling of the transmission of Chlamydia in the United States from 1989 to 2019"** :

The main target is the analysing of future outbreaks of chlamydia in the united states by using the data from 1989 to 2019 . The memory effects in the model which can provide more degree of freedom in the experimental simulations is included in using the Caputo fractional derivatives.

Features of this article: In section (3), the dynamics of integer and fractional order mathematical models along with the novel parameter estimation and data fitting is presented Some computational techniques for estimating the parameter values is used. Firstly an ODE solver in MATLAB for estimating the parameters of the classical model from a Least square perspective is used and it is solved numerically for a set of values for the Caputo fractional derivative. Another technique is used for the Adams- type predictor – correction method to investigate parameter estimation for the fractional- order model. For checking the effects of fractional- order values several experimental observations are provided and the best- fit value of the order of Caputo fractional derivatives is obtained in section 4 of this paper.

An optimal control derivation for the chlamydia modelling is defined and an optimal control modelling of the chronic chlamydia trachomatis disease as a combination treatment with tryptophan and antibiotic is derived. A mathematical modelling of chlamydia trachomatis in a human carrier is introduced and a study on optimizing range for a chlamydia trachomatis screening program is proposed. By using several types of fractional derivatives several real- world problems have been successfully modelled. The existence and uniqueness of a fractional order blood glucose- insulin minimal model is implemented as well as the analysis of the transmission of COVID-19 epidemic by using integer and fractional derivatives. The Atangana- Baleanu operators on a 3D Hopfield neural network model is implemented. An analytic solution of a non- linear multi- order fractional boundary value problem working in chemical reactor theory has been derived. The mathematical models with fractional derivatives for the dye removal adsorption process have been proposed. With the idea of the mathematical modelling and the features of fractional derivatives, a nonlinear model for the chlamydia epidemic in terms of integer and fractional order derivatives is presented.

Mathematical models:

For defining the dynamics of the chlamydia epidemic in integer- and fractional-order derivatives the mathematical models are introduced.

(1) **Integer- order model:** The structure of sexually transmitted disease chlamydia in the United States by using the mathematical model derivation of odionyenma et al. In this six compartments – unvaccinated susceptible individuals S(t), vaccinated susceptible individuals V(t), exposed peoples E(t), infectious peoples I(t), treated humans T(t), and humans with recovery R(t).

Total population size is

$$N(t) = S(t) + V(t) + E(t) + I(t) + T(t) + R(t)$$

Model dynamics is

$$\frac{dS}{dt} = (1\text{-}\phi)\Lambda - \frac{\beta S(I+ \xi T)}{N} + \omega V - \mu S,$$

$$\frac{dV}{dt} = \phi\Lambda - V(\omega + \mu) - \frac{\beta(1-\pi)V(\xi T+I)}{N} - (\sigma + \mu)E + (1\text{-} \rho)\theta T + \frac{\epsilon\beta(I+\xi\xi T)R}{N}$$

$$\frac{dI}{dt} = \sigma E -(\gamma + \eta + \mu + \delta_1)\mathrm{I} + \rho\theta T$$

$$\frac{dT}{dt} = \eta I - (\mu + \theta + \delta_2 + \tau)T$$

$$\frac{dR}{dt} = \gamma I - \frac{\epsilon\beta R(\xi T+I)}{N} + T\tau - \mu R \qquad (1)$$

Where corresponding model parameters

Λ = the birth rate,

β= the transmission rate from S to I class,

ϵ = the reinfection rte,

ρ = the frction of humns of failed treatment

θ = the failure rate of treatment

μ = natural mortality rate of the human populations,

π = the effectiveness of vaccine ,

ϕ = the fraction of recruited humans,

δ_1 = the disease – induced mortality rate for infectious peoples,

δ_2 = the disease- induced death rate of treated humans,

σ = the transmission rate from E to I class,

ξ = the modification parameter,

ω = waning of vaccine rate,

η = the treatment rate for infections individuals ,

γ = the recovery rate of infectious class,

τ = the recovery of treated class.

Data fitting and parameter estimation:

For parameter estimation, an ODE solution in MATLAB for estimating the parameters of the classical model (1) from a Least square perspective is used. The real data used in the estimation is provided at CDC website.

The parameter estimation and data fitting of the model 1 with yearly infected cases of United States is studied from 1989 to 2019 and the Levenberg- Marquardt algorithm with Isqcurvefit function in MATLAB to estimate the parameter values is applied. The integer – order derivatives are free from the memory in the system for being local in nature. So for securing memory effects in the proposed model further analysis using Caputo-fractional derivatives is done.

(2).Fractional-order derivatives:

The integer –order derivatives is generalized by using Caputo fractional derivatives. The fractional- order model for defining the chlamydia epidemic is given as

$${}^{C}D^{\alpha}_{t_{0+}}S(t) = (1-\phi)\Lambda^{\alpha} - \frac{\beta^{\alpha}S(I+\xi T)}{N} - \mu^{\alpha}S + \omega^{\alpha}V,$$

$${}^{C}D^{\alpha}_{t_{0+}}V(t) = \phi\Lambda^{\alpha} - V(\mu^{\alpha} + \omega^{\alpha}) - \frac{(1-\pi)\beta^{\alpha}V(I+\xi T)}{N},$$

$${}^{C}D^{\alpha}_{t_{0+}}E(t) = \frac{(1-\pi)\beta^{\alpha}V(I+\xi T)}{N} + \frac{\beta^{\alpha}S(I+\xi T)}{N} - (\mu^{\alpha} + \sigma^{\alpha})E + (1-p)\theta^{\alpha}T + \frac{\epsilon^{\alpha}\beta^{\alpha}(I+\xi T)R}{N},$$

$${}^{C}D^{\alpha}_{t_{0+}}I(t) = \sigma^{\alpha}E - I(\mu^{\alpha} + \eta^{\alpha} + \gamma^{\alpha} + \delta_1^{\alpha}) + p\theta^{\alpha}T,$$

$^{C}D_{t_{0+}}^{\alpha}T(t) = \eta^{\alpha}I - T(\mu^{\alpha} + \theta^{\alpha} + \tau^{\alpha} + \delta_2^{\alpha})$,

$$^{C}D_{t_{0+}}^{\alpha}R(t) = \gamma^{\alpha}I - \mu^{\alpha}R - \frac{\epsilon^{\alpha}\beta^{\alpha}(I+\xi T)R}{N} + T\tau^{\alpha} \tag{2}$$

Where $0 < \alpha \leq 1$ is the order of the considered fractional derivative $^{C}D_{t_{0+}}^{\alpha}$

Disease- free equilibrium E^* of the proposed fractional –ordered model is

$$E_1(S^*, V^*, E^*, I^*, T^*, R^*) = \left(\frac{(\mu^{\alpha} + \omega^{\alpha})(1-\phi)\Lambda^{\alpha} + \phi\Lambda^{\alpha}\omega^{\alpha}}{(\mu^{\alpha} + \omega^{\alpha})\mu^{\alpha}}, \frac{\phi\Lambda^{\alpha}}{\mu^{\alpha} + \omega^{\alpha}}, 0,0,0,0\right).$$

The estimated basic reproductive number R_0 is defined by

$$R_0 = \frac{\sigma^{\alpha}(\xi\beta^{\alpha}\eta^{\alpha}A + A\beta^{\alpha}\mathcal{Z}_4)}{N^*[\mathcal{Z}_4\ \mu^{\alpha}(\gamma^{\alpha} + \mu^{\alpha} + \delta_1^{\alpha}) + (\delta_2^{\alpha} + \mu^{\alpha} + \tau^{\alpha})(\eta^{\alpha}\sigma^{\alpha} + \eta^{\alpha}\mu^{\alpha}) + \eta^{\alpha}\mu^{\alpha}\theta^{\alpha}(1-p)]}$$

Where $A = (S^* + (1-\pi)V^*)$, $\mathcal{Z}_1 = \omega^{\alpha} + \mu^{\alpha}$, $\mathcal{Z}_2 = \sigma^{\alpha} + \mu^{\alpha}$, $\mathcal{Z}_3 = \mu^{\alpha} + \eta^{\alpha} + \gamma_1^{\alpha} + \delta_1^{\alpha}$

$\mathcal{Z}_4 = \mu^{\alpha} + \theta^{\alpha} + \tau^{\alpha} + \delta_2^{\alpha}$, $N^* = S^* + V^* + E^* + I^* + T^* + R^*$.

In the following table, the estimated parameter values of the Chlamydia system (1) and (2) For different values of α the estimated parameter values of the Chlamydia system (1) and (2) by using Levenberg- Marquardt algorithm and yearly infected cases in United States from 1989 to 2019 is displayed. The parameter values of the integer order model (1) is represented in 2nd column while the 3rd and 4th columns represent the parameter values of the fractional- order model (2). And the 5th column gives the parameters estimated with an optimal value of α.

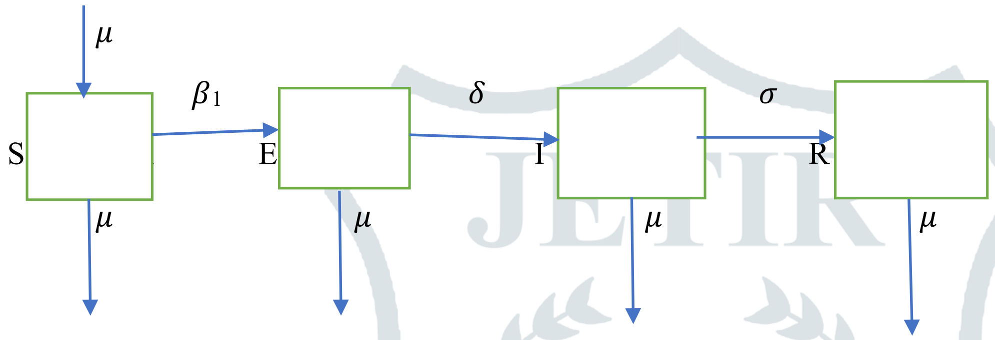
Theorem : The disease-free equilibrium E^* of the proposed fractional –order model (2) is locally asymptotically stable when $R_0 \leq 1$ and unstable when $R_0 > 1$.

Parameter	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.9839$ (Estimated value)
$\emptyset$	3.9491	0.5288	1.7013	1.4284
β	0.8755	1.1089	0.7433	1.0410
ξ	0.7845	0.7624	0.7451	0.9960
ω	0.7139	0.3455	0.4599	0.5117
π	0.4984	0.9999	0.9835	0.000006
σ	0.8988	0.7239	0.9723	0.6348
P	0.6008	0.7269	0.6969	0.000006
ϵ	0.2062	0.6884	0.5521	0.7074
η	0.4629	0.2187	0.1248	0.2489
γ	0.5702	0.6504	0.5293	0.6599
δ_1	0.0928	0.2086	0.000001	0.1526
δ_2	0.0580	0.0128	0.0000034	0.1526
τ	0.5851	0.4456	0.4729	0.7227
θ	0.0020	0.1333	0.3143	0.0297

B. Analysis on [4] "**Disease modelling using Fractional DEs and Estimation**" :

SEIR Model:

To understand how the disease spreads in a population can be understood by mathematical models. The name SEIR model comprises- susceptible, exposed, infections, and recovered components. The flow of individuals from one compartment to the next with the recruitment of new susceptible individuals through birth rate μ is depicted. Like all other compartments, this compartments have a natural death rate μ. The individuals which contact the disease successfully become exposed and then moved to compartment E at transmission rate of βN, when N is the total population size. The explosion after a latent period with average length $\frac{1}{8}$ become infected and thus moving to compartment I at rate δ. The diseases SIR model works best with show symptoms instantly whereas some disease take time to surface. Thus SIR model does not have an exposed compartment as in SEIR model.



Above shows a model displaying the compartmental flow of the SEIR model

The movement from one compartment to another can be expressed through a system of fractional DEs, with $0 < \alpha < 1$.

D_*^{α} S = $\mu - \beta SI - \mu S$, D_*^{α}E = $\beta SI - (\mu + \delta)E$

$D_*^{\alpha} I = \delta$E$-(\mu + \sigma)I$, $D_*^{\alpha} R= \sigma I - \mu R$.

In this the dimension changes when integer-order systems or fractional –order system is used. For integer order systems, $\frac{dI}{dt} = \delta E - (\mu + \sigma)I$ where the dimensions of the parameters $\mu, \delta, \sigma, and\ \beta$ are $\frac{1}{time}$.

When the system is changed into a fractional –order system

D_*^{α}I = $\delta_* E = (\mu_* + \sigma_*)$I.

Simulation and Estimation:

Displaying the effect of $\alpha\ and\ \beta$ on the model is its purpose. Using fractional order, a memory effect is added onto the model that can fit specific diseases and the speed at which a person moved from one compartment to another. Different values of α are compared using increasing values of β^{α} in this paper. The time that a disease one choose to work with moves from one compartment to another may vary depending on the diseases. Different values of β^{α} and α could better …

By integer- order DEs for estimating these models may not fit every disease. But using fractional-ordered equations it is possible to better fit different diseases.

C. **Analysis of the paper [6]" On Literature and Tools in fractional calculus and Applications to mathematical modelling**"

Glimpse of Mathematical Modelling by Fractional calculus:

(i) Diffusion Equation: By applying fractional- order diffusion equation models, anomalous diffusion processes in complex media is symbolized. For the diffusion non locality, the time derivative term depends upon the long-time heavy tail decay and the spatial derivative making the sense for fractional calculus. The time- space fractional diffusion dominating equation is given as

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = - k(-\Delta)^{\beta} u.$$

(ii) Mass conservation Equation:

For modelling fluid flow Wheatcraft and Meers Chaert(2008) proposed a fractional mass conservation equation under the condition of control volume is not large enough as that of the heterogeneity scale and flux's non-linearity. Regarding this, the fractional conservation of mass of mass equation for fluid flow is

$$- \rho(\nabla^{\alpha}.\vec{u}) = \Gamma(\alpha+1)\Delta x^{1-\alpha}\rho(\beta_s + \phi\beta_w)\frac{\partial p}{\partial t}.$$

Model for Immunological response: Several fractional order models are developed for interpreting the immunological response to infection with HIV. A fractional- order differential model of HIV infection of CD4 + T cells consisting of three equations is

$$D^{\alpha^1} H = s - \mu_H H + r H(1- \frac{H+1}{H_{max}}) - k_1 VH,$$

$$D^{\alpha^2} I = k_1^1 VH - \mu_1 I,$$

$$D^{\alpha^3} V = M\mu_b I - k_1 VH - \mu_v V, \quad 0.5 < \alpha_i \leq 1, I = 1,2,3,\ldots\ldots.$$

D. **Analysis of the paper [3] "A new fractional SIRS-SI malaria disease model with application of vaccines, antimalarial drugs, and spraying"**.

A modified mathematical modelling with the assumption that persons of recovered group have the chance to be susceptible. The HAM is applied by Senthamarai et al to inspect the spreading of malaria illness in an SIRS-SI model. Almost approaches and mathematical models being local nature of integer-order derivatives, fractional derivative approaches are proposed in mathematical modelling of biological and physical systems.

Djida and Atangana analysed a water flow inside a confined aquifier connected to an arbitrary order operator in terms of the Caputo- Fabrizio and others. Following the investigations on CF fractional derivative and their effectiveness , this derivative is employed in SIRS-SI malaria model for including memory.

E. In **[5] " Numerical solutions of fractional order Mathematical Model of Drug Resistant Tuberculosis with two line Treatment**":

The Caputo fractional derivative (${}_aD_t^{\alpha}$) of h(t) can be defined as

$${}_aD_t^{\alpha} h(t) = \frac{1}{\Gamma n-\alpha} \int_a^t (t-\tau)^{n-\alpha-1} \; h^{(n)}(\tau) d\tau$$

For n-1 < $\alpha \leq n$ and n ∈ N

t ≥ a ≥ 0 and h ∈ C_{-1}^{n}.

Fractional order mathematical model about drug resistance tuberculosis with two line treatment is discussed in this paper by applying Caputo- fractional derivative. Using generalised Euler method (GEM) this present model is solved. Analysis of mathematical model for giving awareness knowledge in the world is becoming an integral part of this treatment. By utilizing generalized Euler method this fractional order mathematical model of the systems of non-linear ODEs with initial conditions are operated. For the systems of linear and non-linear DE of fractional order classical Euler method is generalized.

If a person is infected and TB treatment is not adequate or transmission is directly from an individual with drug resistant tuberculosis, the first line of treatment is said to be failed and the person may show multi-drug resistance TB (MDR-TB). Extensively drug resistance TB (XDR-TB) may develop with the failure of second line treatment. Then a third type of drug resistance TB which is totally drug resistance TB(XXDR-TB or TDR-TB) is found.

From the analysis the susceptible individuals (S(t)) with recruitment rate in population size N as 'a' have the probability of infected (I(t)). 'c' and 'f' are taken as the rates for an individual shifts from susceptible individuals (S(t)) to exposed individuals (E(t)) and exposed individuals (E(t)) to infected individual (I(t)). Whenever there is infection, the infected person goes for first treatment line (R(t)) or second treatment line (R(t)) with resistance rates to treatment 'h' and 'k'. Then they shift to recovered class (R(t)) after having proper and adequate treatment. 'b' is natural death rate. Taking the rates of disease roused mortality from class I(t), R1(t), and R2(t) as 'g', 'l', and 'n'. It is found that the resistant classes R1(t) and R2(t) on convalescence move to recovered class R(t) at rate 'm' and 'p'.

Sr. No.	Class	Description of class
1	S(t)	Susceptible individuals
2	E(t)	Exposed individuals
3	I(t)	Infected individuals
4	R1(t)	Resistance to first line of treatment.
5	R2(t)	Resistance to second line of treatment
6	R(t)	Recovered individuals.

Standard fractional order mathematical model for transmission of tuberculosis with twin line treatment of Mycobacterium tuberculosis in human host taken as multi drug resistant (MDR) tuberculosis is found to be extended in this work.

The present fractional order mathematical model consist of system of non-linear ordinary differential equations with initial conditions are performed by using generalized Euler method. For the system of non-linear and linear differential equation of fractional order classical Euler method is generalized here.

The following system of fractional order ODE defining the model of Tuberculosis with drug resistant to two line treatment with supposing population as constant (N) are analysed for understanding the directions and mobility of this disease.

$${}_0D_t^{\alpha} S(t) = aN - bs(t) - cS(t)I(t) + dR(t).$$

$${}_0D_t^{\alpha} E(t) = cS(t)I(t) - (b + f)E(t)$$

$${}_0D_t^{\alpha} I(t) = fE(t) - (b + g + h + k)\, I(t)$$

$${}_0D_t^{\alpha} R1(t) = hI(t) - (b + l + m)R1(t)$$

$${}_0D_t^{\alpha} R2(t) = kI(t) - (b + n + p)R2(t)$$

$${}_0D_t^{\alpha} R(t) = mR1(t) + pR2(t) - (b+d)R(t)$$

Where α is real number s.t $0 < \alpha \leq 1$.

The parameters used for the model are

Sr. No.	Class	Description of class
1	a	Recruitment rate in population to susceptible individuals
2	b	Rate of natural death
3	c	Rate at which susceptible individuals be exposed
4	d	Rate at which recovered individuals becomes susceptible again
5	f	Rate at which exposed individuals be infected
6	g	Rate of diseased roused mortality in I(t)
7	h	Resistance to first line of treatment
8	k	Resistance to second line of treatment
9	l	Rate of diseased roused mortality in R1(t)
10	m	Rate of recovery after first line of treatment
11	n	Rate of diseased roused monthly in R2(t)
12	p	Rate of recovery after second line of treatment

Numerical results of this mathematical model:

Fractional ordered mathematical model are solved by applying general Euler method (GEM).By taking initial conditions S(0) = 1, E(0) = 2, I(0) = 1, R1(0) = 1, R2(0) = 1, R(0) =1 the graphical results for $\alpha = 1$ are investigated. Susceptible population increases up the limit point and then takes the steady state. As time increases, the other remaining classes of population decrease and take constant value. From this it can be observed that both first and second line of treatments are found useful for controlling Mycobacterium tuberculosis.The parameters and initial conditions in this model are kept fixed. By changing the order of fractional differential equations numerical simulations of the model are discussed to interpret the change in varied classes. This enables the alternation in order produces a modification in nature of graph for investigation. By 2-D graph in two types of initial conditions and parameter values can illustrate the numerical solution for this present fractional order mathematical model. It is seen that in all the cases, there is gradual variation in asymptotic nature of the classes when the order of derivatives changes. Thus analysing the graphs , the graphs for fractional order mathematical model reflect the dynamic nature of classes under the same initial conditions and parameters.

F. On the book [7] **"Methods of mathematical modelling: Fractional Differential Equations**"

This book can be analysed in 13 chapters : The dynamical nature of two ecosystems of 3 species of prey, intermediate predater and top-predater are presented in chapter 1. In such models the classical integer-order derivatives are replaced with Atangana- Baleanu fractional derivative in Caputo sense. The condition of a dynamical to be locally asymptotically stable is provided in the analysis. A range of chaotic and spatiotemporal phenomena are obtained for different instances of $\alpha \in (0,1)$. In chapter (2), the system is defined in a semiinfinity medium as the solutions for fractional diffusion equations subjected to ……..boundary conditions is investigated. Chapter (3) presents an effective computational method for the approximate solution of the non-linear fractional Lienard equation (FLE) describing the oscillatory circuit and the fractional derivative is in a Liouville- Caputo sense. Combination of collocation method and operational matrix method for Legendre scaling functions is the computational method. In chapter (4), a new approximation scheme for solving fractional DEs with Gomez- Atangana- Caputo derivatives is introduced . Exact solutions or numerical solutions so obtained are compared with other methods. Chapter (5) presents a spectral formulation for a fractional optimal control problem (FOCP) in spherical coordinates. Representing the state and control functions of the systems in numerical methodology is the first step. By using the fractional Euler – Lagrange equations the required optimality conditions are determined in the next step. In the last step by using the Grunuwald- Letnikov approach , the time domain is discretized into a number of subintervals. In chapter (6), a new approximate solution for the fractional diffusion equation given by the Caputo- generalized fractional derivative. In this the approximate solution of the fractional diffusion equation of integer order derivative, the Caputo derivative and Caputo- generalized fractional derivative are discussed and compared. Chapter (7) describes an analytical algorithm for non-linear time fractional Toda lattice equations with the fractional derivatives are in Caputo sense. In chapter (8), the concept of the fractional derivative to the heat transfer problem of a hybrid nanofluid is used. This chapter is dealing with the generalization of natural convection flow of $Cu- Al_2 O_3- H_2 O$ hybrid nanofluid in two infinite vertical parallel plates. In chapter (9), mathematical model from groundwater is solved and is recharged by rain water or spreading the water on the ground in vertical direction. In chapter (10), a range of chaotic and hyperchaotic processes modelled with the Atangana- Baleanu fractional derivative having non-local and non- singular properties in Caputo sense is studied. In chapter (11), a new numerical method , Adomian decomposition Sumudu transform method (ADSTM) is proposed for finding the numerical solution of non-linear time- fractional Zakharov- kuznetsov (FZK) equation in 2-d . In chapter 12, the propagation of wave envelop with fractional temporal evolution by taking the transverse surface in the non-linear dynamic system is studied. Chapter (13) analyses the non-local fractional integro- DEs with finite delay where the linear operator as non-densely described in Banach space.

G. On thesis paper **[2] "A study of mathematical models through system of fractional differential equations and its solutions"** :

A new derivative with non local and no singular kernel followed by some related theoretical and applied result is proposed by the Caputo sense. It is found that some useful properties of the new derivative are used and applied for solving the fractional heat transfer model in Abdon and Dumitru (2016). Also the performance between systems of ordinary and fractional DEs of Caputo fractional derivative sense are compared for depicting the mathematical models of diseases. In this paper a model of string cosmology is constructed for geometric string investigated by Pawar et. Al. (2011) in presence of massless scalar field in modified theory of general relativity.

For investigating the true origin of system of ordinary and fractional differential equations of Caputo sense depicting the mathematical models of diseases, it approaches as mean-field approximations of integer and fractional stochastic process, fractional linear birth and death processes.

H. On [9] **" Dynamics of a fractional order mathematical model for COVID-19 epidemic**"

A new mathematical model for COVID-19 epidemic is formulated and analysed in this work. A system of fractional- order DEs model explains this model and five classes such as S (susceptible class), E(exposed class), I(Infected class), Q (isolated class) , and R (recovered class) are included in this model which is the generaliation of ODE model formulated in " Mathematical model for Coronavirus disease 2019 (COVID-19) containing isolation class" by Zeb, A., Alzahrani et al.(2020)

Aim of this work: To study the dynamics and numerical approximations for this proposed fractional –order model is the objective.

The positivity and boundedness of this considered model are investigated on standard techniques based of mathematical analysis. Secondly by using the next generation matrix approach the basic reproduction number of the model is estimated. After this, asymptotic stability of the model is reviewed. Based on the Lyapunov stability theorem for the fractional dynamical system. Lastly , to define the effectiveness of the theoretical results the adaptive predicter- correcter algorithm and fourth –order Runge- Kutta (RK4) method is applied.

In this work, the fractional – order differential model is formulated in section (2) while dynamics of the model is investigated in section (3). Numerical simulations by the adaptive predicter – correcter algorithm are implemented in section (4).

1.Model formation:

In this model, formulation the whole population is divided into 5 compartments—susceptible (S), exposed (E), infected (I), isolated (Q), and recovered (R) from the disease. In this model the disease death rate is included in the natural death rate. The exposed class moves with a certain rate to the isolated class whenever there is no disease symptoms and it moves to the infected class when symptoms are developed .Based on these assumptions, the ODE model are obtained as

$$\begin{aligned}
\frac{dS(t)}{dt} &= \wedge - \mu S(t) - \beta S(t)\big(E(t)+I(t)\big),\\
\frac{dE(t)}{dt} &= \beta S(t)(E(t)+I(t)) - \pi E(t) - (\mu+\gamma)E(t),\\
\frac{dI(t)}{dt} &= \pi E(t) - \sigma I(t) - \mu I(t), \qquad (1)\\
\frac{dQ(t)}{dt} &= \gamma E(t) + \sigma I(t) - \theta Q(t) - \mu Q(t),\\
\frac{dR(t)}{dt} &= \theta Q(t) - \mu R(t).
\end{aligned}$$

The parameters and variables used are described in the following table as

Symbols used	Description
S	Susceptible population
E	Exposed population
I	Infected population
Q	Isolated population
R	Recovered population
$\wedge = \mu N$	Recruitment rate
β	Rate at which susceptible move to infected and exposed class
π	Rate at which exposed population moves to infected one
γ	Rae at which exposed people become isolated
σ	Rate at which infected people are added to isolated individuals
θ	Rate at which isolated persons become recovered
μ	Natural death rate plus disease related death rate

Where $N = S + E + I + Q + R$ (2)

From (1), $\frac{dN}{dt} = \wedge - \mu N$

The following system of fractional differential equations are found to purpose:

$$ {}_{0}^{C}D_{t}^{\alpha}S = \Lambda^{\alpha} - \mu^{\alpha}S - \beta^{\alpha}S(E+I), $$

$$ {}_{0}^{C}D_{t}^{\alpha}E = \beta^{\alpha}S(E+I) - \pi^{\alpha}E - (\mu^{\alpha} + \gamma^{\alpha})E, $$

$$ {}_{0}^{C}D_{t}^{\alpha}I = \pi^{\alpha}E - \sigma^{\alpha}I - \mu^{\alpha}I, \quad (3) $$

$$ {}_{0}^{C}D_{t}^{\alpha}Q = \gamma^{\alpha}E + \sigma^{\alpha}I - \theta^{\alpha}Q - \mu^{\alpha}Q, $$

$$ {}_{0}^{C}D_{t}^{\alpha}R = \theta^{\alpha}Q(t) - \mu^{\alpha}R(t), $$

where $0 < \alpha < 1$, and ${}_{0}^{C}D_{t}^{\alpha}$ = the fractional derivative in the Caputo sense.

2. Dynamics of the fractional –order model:

2.1 Positivity and boundedness

$$ \mathbb{R}_{+}^{4} = \{(S,E,I,Q) \mid S,E,I,Q \geq 0\}. $$

Theorem 1(Positivity and boundedness): If (S_0, E_0, I_0, Q_0) be any initial data belonging to $\mathbb{R}_{+}^{4}$ and (S(t), E(t), I(t), Q(t)) be the solution corresponding to the initial data, then the set $\mathbb{R}_{+}^{4}$ is a positively invariant set of the above model (3). Furthermore,

$$ \lim_{t\to\infty} sup S(t) \leq S_{\infty} := \frac{\Lambda^{\alpha}}{\mu^{\alpha}}, $$

$$ \lim_{t\to\infty} sup E(t) \leq E_{\infty} := \frac{\Lambda^{\alpha}}{\pi^{\alpha}+\mu^{\alpha}+\gamma^{\alpha}}, $$

$$ \lim_{t\to\infty} sup I(t) \leq I_{\infty} := \frac{\pi^{\alpha}E_{\infty}}{\sigma^{\alpha}+\mu^{\alpha}}, $$

$$ \lim_{t\to\infty} sup Q(t) \leq Q_{\infty} := \frac{\gamma^{\alpha}E_{\infty} + \sigma^{\alpha}I_{\infty}}{\delta^{\alpha}+\mu^{\alpha}}, $$

$$ \lim_{t\to\infty} sup R(t) \leq \frac{\theta^{\alpha}}{\mu^{\alpha}}. $$

By using generalized mean value theorem and a fractional comparision principle, this theorem is found to be proved.

Above model (3) is found to be rewritten in the matrix form as

$$ {}_{0}^{C}D_{t}^{\alpha}x = \mathcal{F}(x) - \mathcal{V}(x) \text{ with their notations in this work.} \quad (4) $$

Theorem 2 (Equilibria): The above model (3) always possesses a disease –free equilibrium (DFE) point F_0 = $(S_0, E_0, I_0, Q_0, R_0)$ for all values of the parameters. Also, the model has a unique disease endemic equilibrium point $F^* = (S^*, E^*, I^*, Q^*, R^*)$ given by (4) iff $\mathcal{R}_0 > 1$.

2.2 Stability analysis

Theorem (3): DFE point of the model (3) is locally asymptotically stable if $\mathcal{R}_0 < 1$.

3. Numerical simulations by the adaptive predictor- corrector algorithm :

3.1 The adaptive predictor-corrector algorithm

The method proposed by Odibat , Z., Baleanu, D. in " Numerical simulation of initial value problems with generalized Caputo-type fractional derivatives " is reviewed.

This proposed algorithm as well as the initial value problem (IVP) are given as

$$\begin{cases} D^{\alpha,\rho}_{\alpha+} y(t) = f(t, y(t)), \ t \in [0, T], \\ y^k(a) = y^k_{0,} \quad k = 0,1,2,3, \dots, [\alpha] - 1, \end{cases}$$

Where $D^{\alpha,\rho}_{\alpha+}$ is the proposed generalized Caputo- type fractional derivative operator.

I. By [1] " Review of fractional epidemic models" , The results of epidemiological modelling mainly for the fractional epidemic model is reviewed and various types of fractional epidemic models such as fractional Susceptible Infective- Recovered (SIR), Susceptible- Exposed- Infective- Recovered (SEIR), Susceptible-Exposed-Infective- Asymptomatic- Recovered (SEIAR) model are discussed. The basic mathematical model describing the expansion infectious diseases is called compartmental model and there are three separate compartments – susceptible compartments S, the infected compartment I, and the removed compartment R. The susceptible person after infection inclines on infected person. This person get contact with an infected person and this infected person is recovered by treatment in the system. The individuals in this class gets permanent immunity for the applicable disease. Such type of model is called SIR model and their variants are studied and also used for examining particular disease like dengue and leptospirosis epidemics.

The fractional order epidemic system with the fractional derivatives explained in section 2 are observed in this paper. In section 3, epidemiology modelling by formulating review is initiated. In section 4 and 5, the corresponding inverse problem for parameter estimation for the fractional differential equation by the novel techniques and optimization algorithm are analysed based on the numerical solution for the model review. Then the Norovirus infectious as an example about application to parameters estimation of fractional SEIAR model is taken in section (6) and for justifying the effectiveness and accuracy of the proposed methods in dealing with the fractional inverse problem a general multi-term fractional- order epidemic system with the new fractional orders and parameters is suggested.

1.**Model Formation Review:**

1.1 Mathematical modelling of dangue epidemic (SIR-SI MODEL)

Dangue fever is a disease creating problems with magnitudes increases dramatically in last 2 decades. Through the bite of infected Aedes mosquitoes, dengue virus is transmitted to humans. A mosquito remains infected for life if it is once infected, transmitting the virus to susceptible individuals during feed. Stability of the classic dangue disease mode is studied by Rodrigues et al. A mathematical model of dangue transmission with memory was proposed by Sardar. The fractional –order dangue system with Riemann- Liouville-type derivatives of the same order was analysed by Pooseh. A fractional- order model based on the Caputo-type derivative for the simulation of an explosion of dangue fever is proposed by Diethelm.

The researchers in 2019 proposed a new and general fractional-order dangue fever using Caputo derivative of different orders with ${}^{C}_{0}D^{\alpha_i}_{t}$ = Caputo fractional derivative with order α_i

For the better dangue fever system in research which can provide numerical results for better results with the real data, a multi-term fractional order dangue model is presented as

$${}^{C}_{0}D^{\alpha_{i,\dots,}\alpha_r,\alpha_0}_{t} S_h = \mu_h(N_h - S_h) - \frac{\beta_h b}{N_h+m} S_h I_m,$$

$${}^{C}_{0}D^{\alpha_{i,\dots,}\alpha_r,\alpha_0}_{t} I_h = \frac{\beta_h b}{N_h+m} S_h I_m - (\mu_h + \gamma) I_h,$$

$${}^{C}_{0}D^{\alpha_{i,\dots,}\alpha_r,\alpha_0}_{t} R_h = \gamma I_h - \mu_h R_h ,$$

$${}^{C}_{0}D^{\beta_{i,\dots,}\beta_r,\beta_0}_{t} S_m = \mu_m(N_m - S_m) - \frac{\beta_m b}{N_h} S_m I_h,$$

$${}^{C}_{0}D^{\beta_{i,\dots,}\beta_r,\beta_0}_{t} I_m = \frac{\beta_h b}{N_h+m} S_m I_h - \mu_m I_m,$$

Where ${}_{0}^{C}D_{t}^{\alpha_{i,\dots,}\alpha_{r},\alpha_{0}}$ and ${}_{0}^{C}D_{t}^{\beta_{i,\dots,}\beta_{r},\beta_{0}}$ are defined as

$${}_{0}^{C}D_{t}^{\alpha_{i,\dots,}\alpha_{r},\alpha_{0}}x(t) = \sum_{i=1}^{r}\lambda_{i}\cdot{}_{0}D_{t}^{\alpha_{i}}x(t) + \lambda_{0}\cdot{}_{0}D_{t}^{\alpha_{0}}x(t),$$

and $${}_{0}^{C}D_{t}^{\beta_{i,\dots,}\beta_{r},\beta_{0}}x(t) = \sum_{i=1}^{r}\lambda_{i}'\cdot{}_{0}D_{t}^{\beta_{i}}x(t) + \lambda_{0}'\cdot{}_{0}D_{t}^{\beta_{0}}x(t),$$

with $0 < \alpha_1 < \cdots .. < \alpha_r < \alpha_0 = 1$ and $0 < \beta < \cdots .. < \beta_r < \beta_0 = 1$.

On the analysis of the "fractional order dengue transmission model: a case study in Malaysia, Advances in Difference Equations" (2019) by N. Hamdan, A. Kilicman, applying the SIR-SI model analysis on a basic fractional order epidemic model of dengue transmission is considered.. The infection is supposed to produce by only one serotype of dengue viruses. In the model formulation process, the total number of human and mosquito population is taken constant. Aquatic phase, Am , and adult mosquito stage are included in the dynamics ( ) of female Aedes mosquito. Again adult stage is grouped into two compartments- susceptible H_s, infectious H_i, and recovered H_r individuals.

Then corresponding system for human population exclusive of the H_r DE is

$${}_{0}^{C}D_{t}^{\alpha}H_s = \mu_h(H - H_s) - \frac{\beta_h b^{\alpha}}{H} H_s M_i,$$

$${}_{0}^{C}D_{t}^{\alpha}H_i = \frac{\beta_h b^{\alpha}}{H} H_s M_i - (\gamma_h + \mu_h) H_i,$$

$${}_{0}^{C}D_{t}^{\alpha}A_m = q\phi(1 - \frac{A_m}{C})M - A_m(\sigma_A + \mu_A),$$

$${}_{0}^{C}D_{t}^{\alpha}M_s = \sigma_A A_m - \frac{\beta_m b^{\alpha}}{H} M_s H_i - \mu_m M_s,$$

$${}_{0}^{C}D_{t}^{\alpha}M_i = \frac{\beta_m b^{\alpha}}{H} M_s H_i \mu_m M_i.$$

In this the values related to the human is describing the real figure of infecyed period in Malaysia.

1.2 Mathematical modelling of Lepospirosis epidemics (**SIR- SI MODEL**):

It is an bacteria infectious disease which flourishes ( ) due to delay in diagnosis and lack of clinical infrastructure. The people having contact ( ) with infected animals , soils, or bacteria present water can easily infected. A mathematical model describing the epidemic leptospirosis disease is proposed ( ) by the researchers as

$$\frac{dS^h(t)}{dt} = b_1 - \mu_h S^h - \beta_2 S^h I^h + \lambda_h R^h,$$

$$\frac{dI^h(t)}{dt} = \beta_2 S^h I^v + \beta_1 S^h I^h - (\mu_h + \delta_h + \lambda_h) I^h,$$

$$\frac{dR^h(t)}{dt} = \gamma_h I^h - \mu_h R^h - \lambda_h R^h,$$

$$\frac{dS^v(t)}{dt} = b_2 - \gamma_v S^v - \beta_3 S^v I^h,$$

$$\frac{dI^v(t)}{dt} = \beta_3 S^v I^h - (\gamma_v + \delta_v) I^v.$$

Where $S^h(t), I^h(t), R^h(t), S^v(t) and\ I^v(t)$ are the population of susceptible human, infected human, recovered human, susceptible vector with infected vector at time t.

1.3 Mathematical modelling of HIV/AIDS epidemics (SIJA MODEL):

Human immunodeficiency virus (HIV) leading to acquired immunodeficiency syndrome (AIDS) is dangerous and fatal if not treated and controlled. Major indicator of the disease stages which correspond to CD4+ T-cell count rnges is the virus number in the blood. The level of CD4+ T-cells of a normal healthy person's peripheral blood is in the range of 800 to 1200/mm^3. A person is treated as having AIDS if once this number reaches 200 or below in an HIV infected patient.

A fractional order model of the transmission dynamics HIV/AIDS with random testing and contact tracing was presented in Cuba , a generalization to fractional order model proposed by Mastroberardino et al.

$${}_0^C D_t^\alpha S(t) = \Lambda - (\varepsilon_1 u_1 \beta + (1-u_1)\beta)XS - \mu S,$$

$${}_0^C D_t^\alpha X(t) = (\varepsilon_1 u_1 \beta + (1-u_1)\beta)XS - kXY - (\mu + \gamma + k')X,$$

$${}_0^C D_t^\alpha Y(t) = \mathrm{kXY} + k' - \big(\mu + (\varepsilon_2 u_2 \gamma + (1-u_2)\gamma)\big)Y,$$

$${}_0^C D_t^\alpha Z(t) = \gamma X + (\varepsilon_2 u_2 \gamma + (1-u_2)\gamma)Y - \mu' Z,$$

Where $0 \le \alpha \le 1$. The model parameters used in this are :

$-\ \Lambda$: Constant recruitment rate of susceptible population;

$-\beta$: recruitment rate of new members of HIV-infected population infected by sexual trans-

mission with X;

$-\gamma$: rate at which HIV- infected population develops AIDS;

$-\kappa$: rate at which undiagnosed HIV- infected population is diagnosed through contact

tracing;

$-\kappa'$: rate at which undiagnosed HIV-infected population is diagnosed through random

testing;

$-\mu$: mortality rate of the adult population;

$-\mu'$: mortality rate of the population with AIDS;

$-u_1$: the proportion of susceptible individuals using condom;

$-u_2$: *the proportion of diagnosed HIV-infected population under ART treatment;*

$-\varepsilon_1$: efficiency of u_1;

$-\varepsilon_2$: efficiency of u_2.

Using the stability theorem and using the fractional La- Salle invariance principle for fractional differential equations (FDEs) the stability of the equilibria of the model are surveyed. By a mass action term the susceptible population are transmitted to the undiagnosed HIV infected population in this model. Using the Adams- type predictor-corrector method, Kheiri and Jafarin developed the Forward- Backward sweep method (FBSM) for numerical simulation. Conditions for fractional optimal control of the disease are obtained and analysed. Also the efficacy of the fractional derivative order α with $3/5 \le \alpha \le 1$ on the HIV/AIDS epidemic model and the controls was examined.

Conclusion: A new mathematical model for COVID-19 epidemic with isolated class helps to understand the mechanism of the transmission , characteristics of diseases and to protect population health. A fractional SIRS-SI malaria model transmission along with a no. of cures is studied. It is predicted that the proposed fractional calculus model provides a wide scope for analysing diseases scientifically. A new mathematical model for COVID-19 epidemic is formulated and analysed described by a system of fractional-order differential equations including five classes. An optimal control system for the mosaic epidemic in terms of Caputo fractional derivatives is derived. The generalized Euler method (GEM) is found to be powerful, effective and reliable method for investigation of fractional mathematical model. The multi- term fractional SEIAR model provides better fit to the real data than the integer order and single term fractional SEIAR models. A non linear mathematical model of the transmission of chlamydia in the United States is simulated.

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