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Conservation through reserve creation

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Abstract

A Mathematical Model is developed to study the effect of toxicant on the growth and survival of biological species in an environment. If the emission rate of toxicant increases without control then species population goes to extinction in the given habitat. It also observed that if suitable efforts are made to reduce the emission rate of toxicant at the source then an appropriate level of species density can be achieved to maintain Ecological balance.

(**Keywords**: control, prey, toxicant, conservation, environment).

1. Introduction

Due to urbanization, industrialization, infrastructure development coupled with human activities cause permanent damage and serious impoverishment for environment. The effluents released from several industries (like fertilizers and chemicals) and agricultural pesticides are main reasons for environmental pollution. The uncontrolled contribution of pollutant to environment has led many species to extinction and several others at verge of extinction.

In the study of Kosysim (2007) on evaluating the pollution status of Moirang river, concluded that the anthropogenic activities, industrial and agricultural effluents released into the river causing damage to aquatic bio – resources in Moirang river (Kosysim, 2007). The plankton population are highly influenced by industrial effluents discharged into the river Cauvery (Mathivanan, 2007).

The study of the industrialization and pollution on biological species using mathematical models and the prediction are very much limited. In recent years some investigations have been conducted to study the effect of pollutant on population using mathematical models (Shukla *et al*., 1989; Hallam *et al*., 1983a, b; Freedman and Shukla, 1991). The mathematical models of single population with toxicant effect have been studied extensively by Mozhier, Hallam (1986) and Srinivasu (2002), Srinivasu, Gayatri (2005), Collings (1995). Hence, the present study was carried out to determine a clean up policy to be implemented at the source of pollution in order to conserve a population in a ecosystem and to develop a model which is non- linear, autonomous system of differential equations with initial conditions.

2. Mathematical Model

The dynamics of single species population - toxicant interaction is considered. It is assumed that the growth of population decreases proportionally to the rate of toxicant entered into the environment. The rate of concentration $C_E(t)$ of toxicant decreases due to natural degradation factors, and consumption of toxicants by population. This rate increases due to egestion of the population. Let x(t) denote the concentration of the population bio mass at a given time t. And this rate increases due to egestion of the population. Let u(t) represents the exogenous rate of toxicant input into the environment and assuming it is a bounded function that is $0 \le u_0 \le u(t) \le u_1 < +\infty$, where u_0 and u_1 are constants.

Thus the following model represents the dynamics of single – species population and toxicant interaction:

$$\frac{dx}{dt} = r\,x\left(1-\frac{x}{k}\right) - \alpha_1 x c_E$$
$$\frac{dC_E}{dt} = -\delta_0 C_E - \beta_1 x C_E + \theta_1 C_E + u(t) \qquad (1)$$

Such that $x(0) = x_0 > 1$ and $C_E(0) = \bar{C}_0 > 0$ are initial conditions and **k>0** is the carrying capacity of prey population satisfying **0<x(o)<x(t)<1**. Here $\alpha_1 > 0$ represents decreasing rate of coefficient due to uptake toxicant by the population , $\delta_0 > 0$ is natural degradation coefficient , $\beta_1 > 0$ the depletion rate coefficient due to consumption by the population and $\theta_1 > 0$ is egestion coefficient of population. It is assumed that there exist T^* representing the maximum level of toxicant concentration in an environment, when prey population survive, but can't grow or reproduce, hence $0 < C_E(0) < C_E(t) < T^*$.

Equilibrium points and stability

The steady states solution of system (1) show the existence of two equilibrium points (0,0) and (K,0) are trivial and axial equilibrium points respectively. The following proposition establishes the existence of positive interior equilibrium.

Proposition I

The system (1) admits a positive equilibrium point if the parameters of system satisfies following properties:

(a) $(\delta_0 - \theta_1) > 0$

(b) $\left[\frac{\delta_0 - \theta_1}{k} - \beta_1\right] > 0$ and

(c) $\left[\delta_0 - \theta_1 - \frac{\alpha_1}{r} u\right] > 0$

Proof: Since the parameter δ_0 natural degradation coefficient of the system (1) is in general large, the properties (a), (b), (c) hold.

The steady state solution are given by

$$\overline{C_E(t)} = \frac{r}{\alpha_1}\left(1 - \frac{\bar{x}}{k}\right) and$$

$$\overline{C_E}(t) = \frac{u(t)}{(\delta_0 - \theta_1) + \beta_1 \bar{x}} \tag{2}$$

$$\bar{x} = \frac{-\left(\frac{\delta_0 - \theta_1}{k} - \beta_1\right) \pm \sqrt{\left(\frac{\delta_0 - \theta_1}{k} - \beta_1\right)^2 + \frac{4\beta_1}{k}\left[\delta_0 - \theta_1 - \frac{\alpha_1}{r} u(t)\right]}}{(2\beta_1 / k)} \tag{3}$$

Equation (3) indicates that there exists a positive value for $\bar{x}$. Substitution of this value in (2) yields the positive equilibrium point $(\bar{x}, \overline{C_E})$.

To study the stability nature of these equilibrium points, consider the Jacobian matrix of the system evaluated at (0, 0) which is given by

$$J(0,0) = \begin{bmatrix} r & 0 \\ 0 & -(\delta_0 - \theta_1) \end{bmatrix} \tag{4}$$

Clearly (0, 0) is hyperbolic equilibrium point with eigen values r and $-(\delta_0 - \theta_1)$. And the Jacobian of the system (1) evaluated at (k, 0) is

$$J(k,o) = \begin{bmatrix} -r & -\alpha_1 k \\ 0 & -(\delta_0 - \theta_1) - \beta_1 k - \lambda \end{bmatrix} \tag{5}$$

The eigen values of the above matrix are the diagonal elements and hence (k,0) is stable node with eigen values – r and $-[(\delta_0 - \theta_1) + \beta_1 k]$. Now evaluating the Jacobian of the considered system at positive equilibrium point $(\bar{x}, \overline{C_E})$,

$$J(\bar{x}, \overline{C_E}) = \begin{bmatrix} r - \frac{2r\bar{x}}{k} - \alpha_1 \overline{C_E} & -\alpha_1 \bar{x} \\ -\beta_1 \overline{C_E} & -[(\delta_0 - \theta_1) - \beta_1 \bar{x}] \end{bmatrix} \tag{6}$$

Now applying (2) in (6), the characteristic equation of above matrix is given as follow.

$$\lambda^2 + \lambda\left[(\delta_0 - \theta_1) + \beta_1 \bar{x} + \frac{r}{k}\bar{x}\right] + \left[r\bar{x}\ \frac{(\delta_0 - \theta_1)}{k} - \beta_1 + \frac{2r}{k}\beta_1 \bar{x}^2\right] = 0 \tag{7}$$

From the properties of (a), (b) and (c) of proposition I, it can be concluded that the eigen values of (7) have negative real part. Therefore the positive equilibrium point $\left(\bar{x}, \overline{C_E}\right)$ the system (1) is asymptotically stable. By applying Du-Lac criterion in Paul (1994), it was found that there are no closed trajectories for the system (1). Hence the required equilibrium point $\left(\bar{x}, \overline{C_E}\right)$ is globally asymptotically stable.

3. Conservation through control on toxicant

To analyze the qualitative nature of the function u(t), which represents the rate of toxicant entered into the environment, let us assume that the rate of toxicant entered into the environment, produced at the source is proportional to consumption of the nature c(t). This consumption nature c (t) can be defined as $\frac{dc}{dt} = -\delta_1 e^{-t}$ with initial condition c (0) = μ, whose solution is in the form of $c(t) = \delta_1[\exp(-t) + \mu]$, where exp(-t) represents the decrease in rate of nature degradation (due to deforestation etc) and $\delta_1 > 0$ represents decrease rate in assimilation nature due to lack of conservation policy . Suppose certain amount of toxicant concentration at the source removed is proportional to the amount of the pollutant produced at this source, then the dynamics of the toxicant (Pollutant) produced at the source is formulated as $\frac{du}{dt} = \delta_1(\exp(-t) + \mu) - \eta u$,with initial Condition u(0) = u_0 and η>0 represents clean up rate at the source, where u_0 is a very small (negligible) quantity of emission released from source initially. Then the solution of the above differential equation is

$$u(t) = \frac{\delta_1}{\eta - 1}\left[e^{-t} - e^{-\eta t}\right] + \frac{\delta_1 \mu}{\eta}\left[1 - e^{-\eta t}\right] + u_0 e^{-\eta t} \qquad (8)$$

The above solution u(t) is bounded and $0 < u_0 \le u(t) \le u_1 < \infty$.

where $u_1 = \frac{\delta_1 \mu}{\eta}$. Equation (8) gives the rate of toxicant emission or discharge entering into environment for conservation (or persistent) of population.

4. Dynamic Optimal Control

Optimization technique is applied to control on u(t) (the rate of toxicant entering in environment) for conservation of single – species population. Therefore the main objective is to solve the optimal control problem.

Minimizing $$I(x, C_E, u) = \int_{to}^{t_f} u(t)dt, \; 0 < t_0 < t_f < \infty$$

With differential equation: $$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \alpha_1 x C_E$$

$$\frac{dC_E}{dt} = -\delta_0 C_E - \beta_1 x C_E + \theta_1 C_E + u(t)$$

With constraint $x(0)= x_0$ and $C_E(0) = \overline{C_0}$ and $0 \le u_0 \le u(t) < u_1 < \infty \; for \; t \in [0, \infty)$.

The optimal control problem is to find an admissible control function u(t) , which drives the system from observed state $(x(0), C_E(0))$ to $P(\overline{x}, \overline{C_E})$ in such a way that the performance index I is minimized.

Applying the Lagrange's theorem of optimal control in Cesari (1983), the Hamiltonian function for the problem is given by :

H (x, C_E, u $\lambda_0, \lambda_1, \lambda_2$)

$$= \lambda_0 u + \lambda_1 \left[r\left(1 - \frac{x}{k}\right)x - \alpha_1 x C_E \right] +$$

$$\lambda_2 \left[-\delta_o C_E - \beta_1 x C_E + \theta_1 C_E + u(t) \right] \quad (9)$$

For optimal control $u=u^*(t)$ and trajectory $(x^*(t), C_E^*(t))$, $t_0 \le t \le t_f$ and $\overline{\lambda} = (\lambda_0, \lambda_1, \lambda_2)$, where $\lambda_0, \lambda_1, \lambda_2$ are costate (adjoint variables), not all zero which satisfy λ_0 = constant > 0 and $\dot{\lambda}_1 = \frac{-\partial H}{\partial x} = \lambda_1 \left[r - \frac{2rx}{k} - \alpha_1 C_E \right] + \lambda_2 \left[-\beta_1 C_E \right]$

(10)

$$\dot{\lambda}_2 = \frac{-\partial H}{\partial C_E} = \lambda_1 \left[(-\alpha_1 x) + \lambda_2 (-\delta_0 - \beta_1 x + \theta_1) \right] \quad (11)$$

Furthermore $\lambda_0, \lambda_1, \lambda_2$ (12)

and along an optimal trajectory, the optimal control u^* (t) minimizes

$H(x^*(t), C_E^*(t), u^*, \lambda_0, \lambda_1, \lambda_2) = 0$

with respect to all admissible controls.

Hence $H_u > 0 \Rightarrow u^*(t) = 0$ (13)

$H_u < 0 \Rightarrow u^*(t) = u_{max}$ (14)

Where the subscript denotes partial differentiation

If on the subinterval (t_1, t_2)

$H_u = 0$ $\Rightarrow \lambda_0 + \lambda_2 = \mathbf{0}$

$\Rightarrow \lambda_0 = -\lambda_2$ (15)

Then optimal problem have a singular control. This is because u appears linearly in the Hamiltonian. Now the total differentiation of (15) with respect to time t, then the result is

$$DH_u = \dot{\lambda}_2 = 0 \qquad (16)$$

Employing (15) and (11) in (16),

$$DH_u = \lambda_1 \alpha_1 x - \lambda_0 \left(\delta_0 + \beta_1 x - \theta_1\right) = 0 \qquad (17)$$

this implies

$$\lambda_1 = \frac{\lambda_0\left(\delta_0 + \beta_1 x + \theta_1\right)}{\alpha_1 x} \quad \text{for} \quad \alpha_1 > 0, x > o \qquad (18)$$

Similarly

$D^2H_u=0$ implies

$$\lambda_0\left\{\left(\frac{\delta_0 - \theta_1}{k}\right)\overline{x} - \left(r - \frac{2rx}{k} - \alpha_1 C_E\right)\left(\delta_0 + \beta_1 - \theta_1\right) - \beta_1 \alpha_1 C_E x\right\} = 0$$

Which implies

$\lambda_0 = 0,$ hence $\lambda_1 = 0,$ and $\lambda_2 = 0$ [from 15 to 17].

Which is not possible .

Hence u is not singular. Therefore u takes the values u = 0 and $u=u_{max}$. The optimal solution of the system lies in between values u = 0 and $u= u_{max}$.

5. Numerical Simulation

In this section, the important results developed in the previous sections are verified for the values of the parameters of the system (5) given in table (1). Since u(t) is bounded and takes the value between $u^*(t) = 0$ and $u^*(t) = u_{max}$, the changes in dynamics of this system for different values of u(t) in these intervals are calculated numerically. Considering the initial values of the defined system as $(x(0), T(0)) = (17, 0.3)$ the positive equilibrium point takes the values according to different values of u(t). From the numerical analysis it is observed that when u(t) takes small values in the interval (u_{min}, u_{max}), prey population grow and reaches optimum level. As u(t) takes value u_{max} or more, the population goes on extinction. These results are evident in Figure 1. Thus as long as the exogenous rate of toxicant is small the population persist and when higher level of toxicant enters into the environment the population goes on extinction.

Table 1.

	k	α_1	δ_0	β_1	θ_1	δ	η	μ

.01	2	.003	.8	.007	.001	.01	.2	.5

Since the function u(t) is bounded and we have

$u_{\min} = 0 \le u(t) \le \dfrac{\delta\mu}{\eta} = u_{\max}$, for the values of table (1), varies from 0 to 4.166.

Figure 1.

Conclusion

This paper deals with a model representing the dynamics of population (single – species) and toxicant interaction in a given ecosystem. It was found that the rate of toxicant level entering into ecosystem influences the existence of single – species population in a given habitat. Therefore in order to conserve a population it is necessary to control u (t) (the exogenous input rate of toxicant). Optimal control method is applied to regulate u(t) and also measures have been taken to reduce the concentration of toxicant at source level for persistence of population up to descried level. From this analysis it can be concluded that an appropriate level of species population can be maintained by controlling the rate of influx of toxicant at source as well as by considering clean–up policy to conserve biological population to maintain ecological balance.

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