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Approximate Solutions of Fractional Radon Diffusion Equation Using Crank-Nicolson Finite Difference Scheme

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Abstract

This study introduces a finite difference numerical technique to simulate the solutions of space-fractional Radon diffusion equation within a water medium. We develop the fractional order Crank-Nicolson finite difference scheme, utilizing space- fractional derivatives in the Caputo sense. We discuss the stability of the solution obtained by the developed Crank-Nicolson finite difference scheme. Furthermore, we delve into the convergence of developed finite difference scheme. Lastly, we represent approximate solutions graphically with the help of Python programs.

Keywords: Radon Diffusion Equation, Crank-Nicolson, Finite Difference Scheme, Sta- bility, Convergence.

1 Introduction

Radon is a naturally occurring noble gas, meaning it is almost chemically inert and unlikely to combine with other atoms to form molecules. It has no color, odor, or taste. Radon is soluble in water, but its solubility decreases with increasing temperature. With a density of 9.73 g/l under standard conditions, it is the heaviest natural gas. When cooled below its freezing point, radon exhibits a brilliant phosphorescence that turns yellow at lower temperatures and orange-red at the temperature of liquid air. Radon is sometimes classified as a metalloid, positioned diagonally between true metals and nonmetals on the periodic table, and shares characteristics with boron, germanium, antimony, and polonium. Radon is absorbed by substances like charcoal and silica gel, which allows for its separation from other gases. It can be effectively removed from an air sample by collecting it on activated charcoal cooled to the temperature of solid CO_2 (-78.5 C). Radon can then be released from the charcoal by heating it to 350 C [1, 2, 3, 4].

In this paper, We present a Crank-Nicolson finite difference scheme to solve space- fractional Radon diffusion equation. We use Grunwald Letnikov fractional derivative for that space fractional differentiation involving in considered equation [5, 9, 10, 11, 12, 13, 14] .

We consider the following one-dimensional space-fractional diffusion equation:

$$\frac{\partial V(x,t)}{\partial t} = D\frac{\partial^\beta V(x,t)}{\partial x^\beta} - \lambda V(x,t), 1 < \beta \le 2, \quad (1)$$

with initial condition:

$$V(x,0) = 0, 0 < x < L \quad (2)$$

and boundary conditions:

$$V(0,t) = V_0 \text{ and } \frac{\partial V(x,t)}{\partial t} = 0, t \ge 0 \quad (3)$$

where, λ is the decay constant, D is a diffusion coefficient, x and t are spatial and temporalvariables respectively.

Definition 1.1. *The Grunwald Letnikov space fractional derivative of order is definedby,*

$$\frac{\partial^\beta V(x,t)}{\partial x^\beta} = {}_0D_x^\beta V(x,t) = \frac{1}{h^\beta}\lim_{N\to\infty}\sum_{j=0}^{N}\frac{\Gamma(j-\beta)}{\Gamma(-\beta)\Gamma(j+1)}V(x-(j-1)h,t)$$

$$= \frac{1}{h^\beta}\lim_{N\to\infty}\sum_{j=0}^{N} g_{\beta,j} V(x-(j-1)h,t)$$

where

$$g_{\beta,j} = \frac{\Gamma(j-\beta)}{\Gamma(-\beta)\Gamma(j+1)}$$

2 Finite Difference Scheme

In this section, we develop the space-fractional Crank-Nicolson finite difference methodfor the space-fractional order radon diffusion equation (1)-(3)[6, 7, 8, 15].

Let $V(\zeta_i,\tau_k), i = 0,1,2,...,M$ and $k = 0,1,2,...,N$ be the exact solution of time frac-tional radon diffusion equation (1)-(3) at the mesh point (ζ_i,τ_k), where $\tau_k = k\tau$, $k = 0,1,2,\ldots,N$ and $\zeta_i = ih$, $i = 0,1,2,\ldots,M$, where $\tau = \frac{T}{N}$ and $h = \frac{L}{M}$. Let V_i^k be the numerical approximation of the point $V(ih,k\tau)$.

We investigate the spatial β-order fractional derivative using the Grunwald finite differ- ence formula at various temporal stages. Conventional Grunwald approximations fre- quently result in unstable finite difference equations, regardless of whether the method is explicit or implicit. Therefore, we employ a right-shifted Grunwald formula to improve the accuracy of the spatial β-order fractional derivative approximation[10, 12, 16].

We have,

$$\frac{\partial^{\beta} V(x,t)}{\partial x^{\beta}} = D^{\beta} V(x,t) = \frac{1}{h^{\beta}} \sum_{j=0}^{i+1} g_{\beta,j} V(x_{i-(j-1)h}, t_{k+1}) + O(h^2)$$

and the normalized Grunwald weights are given by,

$$g_{\beta,0} = 1 \text{ and } g_{\beta,j} = (-1)^j \frac{\beta(\beta-1)\cdots(\beta-j+1)}{j!}, \; j = 1, 2, 3, \cdots$$

Employing the forward difference formula for time and the right-shifted Grunwald formula for second-order space, we derive a Crank-Nicolson type numerical approximation for equation (1), which is expressed as follows:

$$\frac{V_i^{k+1} - V_i^k}{\tau} = \frac{D}{2h^{\beta}} \left[\sum_{j=0}^{i+1} g_{\beta,j} V_{i-j+1}^{k+1} + \sum_{j=0}^{i+1} g_{\beta,j} V_{i-j+1}^{k} \right] - \lambda V_i^{k+1}$$

$$V_i^{k+1} - V_i^k = D \frac{\tau}{2h^{\beta}} \left[\sum_{j=0}^{i+1} g_{\beta,j} V_{i-j+1}^{k+1} + \sum_{j=0}^{i+1} g_{\beta,j} V_{i-j+1}^{k} \right] - \lambda \tau V_i^{k+1}$$

Let $r = D\frac{\tau}{2h^{\beta}}$ and $\mu = \lambda\tau$, we obtain

$$V_i^{k+1} - V_i^k = r \left[\sum_{j=0}^{i+1} g_{\beta,j} V_{i-j+1}^{k+1} + \sum_{j=0}^{i+1} g_{\beta,j} V_{i-j+1}^{k} \right] - \mu V_i^{k+1}$$

$$(1+\mu) V_i^{k+1} - r \sum_{j=0}^{i+1} g_{\beta,j} V_{i-j+1}^{k+1} = V_i^k + r \sum_{j=0}^{i+1} g_{\beta,j} V_{i-j+1}^{k} \quad (4)$$

$$(1+\mu) V_i^{k+1} - r g_{\beta,1} V_i^{k+1} - r \sum_{j=0, j\neq 1}^{i+1} g_{\beta,j} V_{i-j+1}^{k+1} = V_i^k + r g_{\beta,1} V_i^k + r \sum_{j=0, j\neq 1}^{i+1} g_{\beta,j} V_{i-j+1}^{k}$$

since $g_{\beta,1} = -\beta$

$$(1+\mu+r\beta) V_i^{k+1} - r \sum_{j=0, j\neq 1}^{i+1} g_{\beta,j} V_{i-j+1}^{k+1} = (1-r\beta) V_i^k + r \sum_{j=0, j\neq 1}^{i+1} g_{\beta,j} V_{i-j+1}^{k}$$

Therefore, the complete discretized problem is: For $k = 0$,

$$(1 + r\beta + \mu)V_i^{1} - r\sum_{j=0, j\neq 1}^{i+1} g_{\beta,j} V_{i-j+1}^{k+1} = (1 - r\beta)V_i^{0} + r\sum_{j=0, j\neq 1}^{i+1} g_{\beta,j} V_{i-j+1}^{0}$$

For $k \geq 1$,

$$(1 + r\beta + \mu)V_i^{k+1} - r\sum_{j=0, j=1}^{i+1} g_{\beta,j} V_{i-j+1}^{k+1} = (1 - r\beta)V_i^{k} + r\sum_{j=0, j=1}^{i+1} g_{\beta,j} V_{i-j+1}^{k}$$

initial conditions,

$$V_i^0,\ i = 0, 1, 2, \cdots, M$$

boundary conditions, $V_0^k = V_0$ and

$$V_{M+1}^k = V_{M-1}^k \quad ; k = 0, 1, 2, \cdots, N$$

and $r = D\frac{\tau}{2h^{\beta}}$; $\mu = \lambda\tau$.

The matrix representation of the aforementioned initial boundary value problem is as follows:

$$AV^1 = BV^k + S;\ for\ k = 0 \qquad (5)$$

$$AV^{k+1} = BV^k + S',\ for\ k \geq 1 \qquad (6)$$

where

$$A = \begin{pmatrix} 1 + r\beta + \mu & -rg_{\beta,0} & \cdots & \cdots & \cdots & \cdots \\ -rg_{\beta,2} & 1 + r\beta + \mu & -rg_{\beta,0} & \cdots & \cdots & \vdots \\ -rg_{\beta,3} & -rg_{\beta,2} & \ddots & -rg_{\beta,0} & & \vdots \\ \vdots & \vdots & & 1 + r\beta + \mu & -rg & \vdots \\ \vdots & \vdots & & & \ddots & \ddots \\ -rg_{\beta,M} & -rg_{\beta,M-1} & \cdots & \cdots & -r(g_{\beta,0} + g_{\beta,3}) & 1 + r\beta + \mu \end{pmatrix}$$

$$B = \begin{pmatrix} (1 - r\beta) & rg_{\beta,0} & \cdots & \cdots & \cdots & \cdots \\ rg_{\beta,2} & (1 - r\beta) & -rg_{\beta,0} & \cdots & \cdots & \vdots \\ -rg_{\beta,3} & -rg_{\beta,2} & 1 + r\beta & -rg_{\beta,0} & \ddots & \vdots \\ \vdots & -rg & & \cdots & \ddots & \vdots \\ \vdots & & & & \ddots & \\ 4g_{\beta,M} & rg_{\beta,M-1} & \cdots & \cdots & r(g_{\beta,0} + g_{\beta,2}) & (1 - r\beta) \end{pmatrix}$$

$$S = \begin{pmatrix} rg_{\beta,2}V_0^1 + rg_{\beta,2}V_0^0 \\ rg_{\beta,3}V_0^1 + rg_{\beta,3}V_0^0 \\ rg_{\beta,4}V_0^1 + rg_{\beta,4}V_0^0 \\ \vdots \\ rg_{\beta,M+1}V_0^1 + rg_{\beta,M+1}V_0^0 \end{pmatrix}; \quad S' = \begin{pmatrix} rg_{\beta,2}V_0^{K+1} + rg_{\beta,2}V_0^k \\ rg_{\beta,3}V_0^{K+1} + rg_{\beta,3}V_0^k \\ rg_{\beta,4}V_0^{K+1} + rg_{\beta,4}V_0^k \\ \vdots \\ rg_{\beta,M+1}V_0^{K+1} + rg_{\beta,M+1}V_0^k \end{pmatrix},$$

$V^k = [V_1^k, V_2^k, V_3^k, \cdots, V_M^k]^T$; $r = \frac{D\tau}{h^\beta}$; $\mu^0 = \lambda\tau$; $i = 0, 1, 2, \cdots, M$; $k = 0, 1, 2, \cdots, N$; $g_{\beta,0} = 1$ and $g_{\beta,j} = (-1)^j \frac{\beta(\beta-1)\cdots(\beta-j+1)}{j!}$, $j = 1, 2, 3, \cdots$

3 Stability

Theorem 3.1. *The solution of approximated initial boundary value problem (2.3)-(2.6) for space fractional radon diffusion equation (SFRDE) (1.3)-(1.5) is unconditionally sta-ble.*

Proof: We assume that, $\|E^k\|_\infty \leq |\epsilon_l^k| = \max_{1\leq i\leq M} |\epsilon_i^k|$

Therefore, for $k = 0$, from equation (2.3), we get

$$|\epsilon_l^1| = \left|(1 + r\beta + \mu)\epsilon_l^1 - r\sum_{j=0, j\neq 1}^{i+1} g_{\beta,j}\,\epsilon_{i-j+1}^1\right|$$

$$= \left|(1 - r\beta)\epsilon^0 + r\sum_{j=0,j=1}^{i+1} g_{\beta,j}\,\epsilon_{i-j+1}^0\right|$$

$$\leq \left|1 - r\beta + r\sum_{j=0,j=1}^{i+1} g_{\beta,j}\right|\epsilon_l^0; \quad \because (g_{\beta,1} = -\beta)$$

$$\leq |\epsilon_l^0|; \quad \because \sum g_{\beta,j} < 0 \Rightarrow 1 + r\sum g_{\beta,j} < 1$$

Therefore,

$$\|E^1\|_\infty \leq \|E^0\|_\infty$$

Thus, the result is true for $k = 0$.

Suppose that, the result is true for k,

$$\|E^k\|_\infty \leq \|E^0\|_\infty$$

To prove that the result is true for k+1

$$|\epsilon_l^{k+1}| = \left|(1 + r\beta + \mu)\epsilon_l^{k+1} - r\sum_{j=0,j=1}^{i+1} g_{\beta,j}\,\epsilon_{i-j+1}^{k+1}\right|$$

$$= (1 - r\beta)\epsilon^k + r \sum_{j=0, j\neq 1}^{i+1} g_{\beta,j}\, \epsilon^k_{i-j+1}$$

$$\leq \left(1 - r\beta + r \sum_{j=0, j\neq 1}^{i+1} g_{\beta,j}\right) \epsilon^0; \quad \because (g_{\beta,1} = -\beta)$$

$$\leq |\epsilon^0|; \quad \because \sum g_{\beta,j} < 0 \Rightarrow 1 + r \sum g_{\beta,j} < 1$$

Therefore,

$$\|E^{k+1}\|_\infty \leq \|E^0\|_\infty$$

Thus, the result is true for $k + 1$.
Hence by mathematical induction, the result is true for all k.

$$\|E^{k+1}\|_\infty \leq \|E^0\|_\infty$$

Thus, the scheme is unconditionally stable.

4 Convergence

In this section, we discuss the convergence of the approximate finite difference scheme (2.3) - (2.6). Let $V(x_i, t_k)$ be the exact solution of the SFRDE (1.3)-(1.5) and c^k_i be the exact solution of the discrete equation (2.3)-(2.6) at the mesh point (x_i, t_k), where $i = 0, 1, \cdots M - 1; k = 1, 2, \cdots N.$
We define, $e^k_i = V(x_i, t_k) - V^k_i$, where $i = 0, 1, \cdots, M - 1; k = 1, 2, \cdots N$ and $E^k = (e^k_1, e^k_2, \cdots, e^k_{m-1})$
Now, we have, $E^0 = 0, E^k_0 = 0$ and $E^k_M = 0.$
From (2.3), we get

$$(1 + r\beta + \mu)e^1_i - r \sum_{j=0, j=1}^{i+1} g_{a,j} e^1_{i-j} = (1 - r\beta) + r \sum_{j=0, j=1}^{i+1} g_{a,j} e^0_{i-j} \quad for\ k = 0 \quad (7)$$

From (2.4), we get,

$$(1 + r\beta + \mu)e^{k+1}_i - r \sum_{j=0, j=1}^{i+1} g_{\beta,j} e^{k+1}_{i-j+1} = (1 - r\beta) + r \sum_{j=0, j\neq 1}^{i+1} g_{\beta,j} e^k_{i-j+1}, \quad for\ k > 1 \quad (8)$$

where $r = D\frac{\tau}{2h^\beta}; \mu = \lambda\tau$

Theorem 4.1. *The fractional order crank-Nicolson finite difference scheme (2.3)-(2.6)for SFRDE (1.3)-(1.5) is convergent and the solution of the discretize scheme (2.3)-(2.6)* *and the solution of the equation (1.3)-(1.5) satisfy,*

$\|V(x_i, t_k) - V_i^k\| \leq \|E\|_\infty + 0(\tau + h^{2-\beta}); i = 0, 1, \cdots, M-1; k = 0, 1, \cdots, N.$

Proof: Let us assume that,

$$|e_l^k| = \max_{1 \leq i \leq M-1} |\epsilon_i^k| = \|E\|_\infty;\ for\ l = 1, 2, \cdots$$

and

$$T_l^k = \max_{1 \leq i \leq N} |T_i^k|;\ T_i^n = h_j^2[0(\tau) + 0(h^{2-\beta})|$$

Therefore, from equation (4.1), we have

$$|e_l^1| = \left|(1 + r\beta + \mu)e_i^1 - r \sum_{j=0, j\neq 1}^{i+1} g_{\beta,j}\, e_{i-j+1}^1\right|$$

$$= \left|(1 - r\beta)e_i^0 + r \sum_{j=0, j=1}^{i+1} g_{\beta,j}\, e_{i-j+1}^0\right|$$

$$\leq \left(1 - r\beta + r \sum_{j=0, j=1}^{i+1} g_{\beta,j}\right) |e_l^0|;\ since\ g_{\beta,1} = -\beta$$

$$\leq |e_l^0|;\quad because \sum g_{\beta,j} < 0 \Rightarrow 1 + r \sum g_{\beta,j} < 1$$

Therefore,

$$\|E^1\|_\infty \leq \|E^0\|_\infty + h^2[0(\tau) + 0(h^{2-\beta})]$$

Suppose that

$$\|E^k\|_\infty \leq \|E^0\|_\infty + h^2[0(\tau) + 0(h^{2-\beta})]$$

From equation (2.4), we have

$$|e_l^{k+1}| = \left|(1 + r\beta + \mu)e_i^{k+1} - r \sum_{j=0, j\neq 1}^{i+1} g_{\beta,j}\, e_{i-j+1}^{k+1}\right|$$

$$= \left|(1 - r\beta)e_i^k + r \sum_{j=0, j\neq 1}^{i+1} g_{\beta,j}\, e_{i-j+1}^k\right|$$

$$\leq \left(1 - r\beta + r \sum_{j=0, j=1}^{i+1} g_{\beta,j}\right) |e_l^k|;\ since\ g_{\beta,1} = -\beta$$

$$\leq |e_l^k|;\quad because \sum g_{\beta,j} < 0 \Rightarrow 1 + r \sum g_{\beta,j} < 1$$

Therefore,

$$\|E^{k+1}\|_\infty \leq \|E^0\|_\infty + h^2[0(\tau) + 0(h^{2-\beta})]$$

Hence by mathematical induction, the result is true for all k.

$$\|E^k\|_\infty \leq \|E^0\|_\infty + h^2[0(\tau) + 0(h^{2-\beta})]$$

This shows that fractional finite difference scheme (2.3)-(2.6) for SFRDE (1.3)-(1.5) isconvergent.

5 Numerical Experiments

We consider the following space-fractional radon diffusion equation

$$\frac{\partial V(x,t)}{\partial t} = D\frac{\partial^\beta V(x,t)}{\partial t^\beta} - \lambda V(x,t),\ 1 < \beta \leq 2, \tag{9}$$

Initial condition: $V(\zeta, 0) = 0,\ 0 < z < L$

Boundary conditions:

$$V(0,\tau) = dcV_0, \text{ and } \quad \frac{\partial V(L,\tau)}{\partial \tau} = 0,\ \tau \geq 0$$

Exact solution for $\gamma = 1$ is as follows,

$$V(\zeta,\tau) = dcV_0\left[\frac{\cosh\sqrt{\frac{\lambda}{D}}(L-\zeta)}{\cosh\sqrt{\frac{\lambda}{D}}L} - D\pi\sum_{n=0}^{\infty}\frac{(2n+1)e^{-\left(\frac{(2n+1)^2\pi^2 D}{4L^2}+\lambda\right)\tau}}{L^2\left(\frac{(2n+1)^2\pi^2 D}{4L^2}+\lambda\right)}\sin\frac{(2n+1)\pi\zeta}{2L}\right]$$

In Table 4.1, the approximate solution of time-fractional radon diffusion equation derived from the fractional finite difference scheme is compared with the exact solution for the parameters $\lambda = 2.1\times10^{-6}$, $T = 1$, $L = 1$, $D = 1\times10^{-9}$, $V(0,\tau) = 1$, which demonstrating the method's effectiveness.

Now, we take another set of particular values for the parameters as follows: Radon diffusivity coefficient in water $D = 1\times10^{-9}Bq/m^3$, Spatial length $L = 1.7278\ cm$, Radon decay constant $\mu = 2.1\times10^{-6}$, Adsorption coefficient $c = 4m^2/kg$, Material density $d = 0.5g/cm^3$ and Constant Radon concentration in air $V_0 = 200Bq/m^3$.

With this parameters, we simulate the radon concentration in water after $\tau = 12\ hrs$ in the Figure 4.1 and we can observe that radon loss in its concentration with length. Also, in Figure 4.3, we present radon concentration for various values of β and observe that radon concentration increases more rapidly as β increases. Figure 4.2 shows the simulation of radon concentration for different time slots and we observe that radon

	Absolute error				
$\rightarrow$ $\downarrow$	0.2	0.4	0.6	0.8	.0
.0	$.03\times 10^{-4}$	$.41\times 10^{-5}$	$.93\times 10^{-5}$	$.34\times 10^{-5}$	$.18\times 10^{-5}$
.2	$.45\times 10^{-5}$	$.44\times 10^{-5}$	$.22\times 10^{-5}$	$.74\times 10^{-5}$	$.61\times 10^{-5}$
.4	$.94\times 10^{-5}$	$.64\times 10^{-5}$	$.65\times 10^{-5}$	$.25\times 10^{-5}$	$.14\times 10^{-5}$
.6	$.69\times 10^{-5}$	$.99\times 10^{-5}$	$.17\times 10^{-5}$	$.85\times 10^{-5}$	$.76\times 10^{-5}$
.8	$.67\times 10^{-5}$	$.45\times 10^{-5}$	$.78\times 10^{-5}$	$.51\times 10^{-5}$	$.44\times 10^{-5}$
.0	$.83\times 10^{-5}$	$.01\times 10^{-5}$	$.46\times 10^{-5}$	$.24\times 10^{-5}$	$.18\times 10^{-5}$

Table 1: table
Absolute error

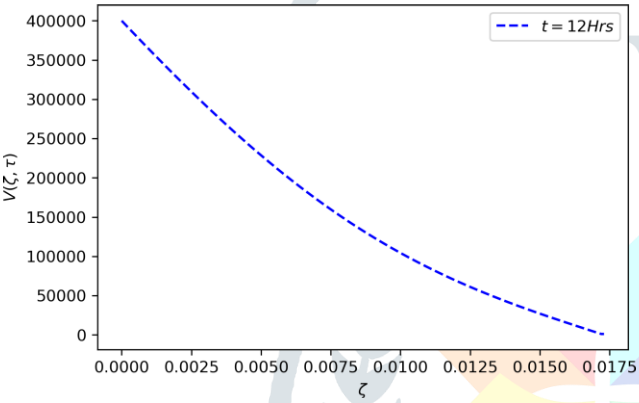

Figure 1: Radon concentration for $\tau = 12\ hrs, \gamma = 1, h = 0.001,$

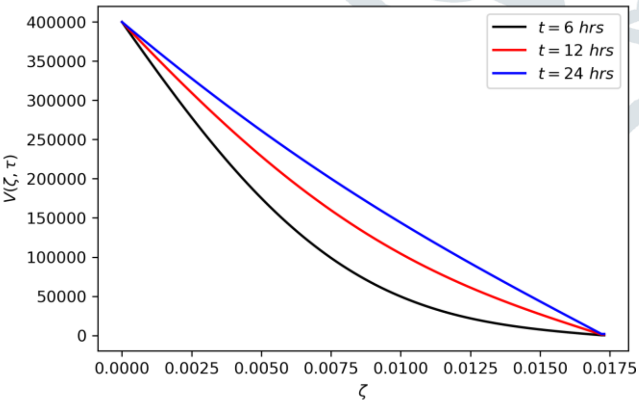

Figure 2: Radon concentration for $t = 12\ hrs, h = 0.001,$

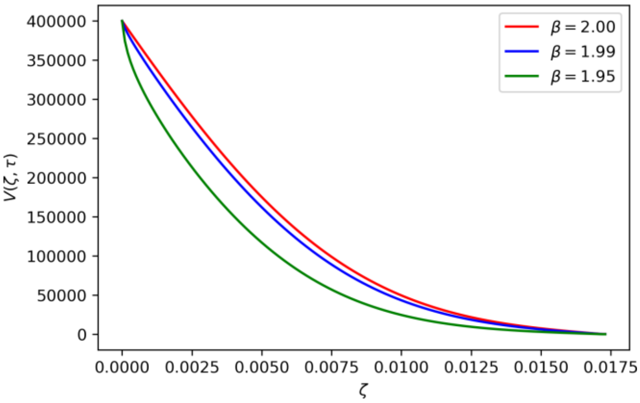

Figure 3: Radon concentration for $\gamma = 1, h = 0.001$,concentration gets increases as time passes.

6 Concluding Remarks

(i) We have successfully developed a fractional-order Crank-Nicolson finite differencescheme tailored for the space-fractional Radon diffusion equation.

(ii) Moreover, we extensively discuss the stability and convergence characteristics of thescheme.

(iii) As an illustrative application of this method, we compute the numerical solution for a given text problem.

(iv) Subsequently, we visualize the obtained solution through graphical simulation im- plemented using a Python program.

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