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Approximation of Function in the Weighted Zygmund class by Matrix – Cesaro Summability Means of Fourier Series

Santosh Kumar Sinha

Deptt of Mathematics, Lakhmi Chand Institute of Technology

Bilaspur (C.G.) India.

Abstract – In this paper, a theorem on degree of approximation of function in the weighted Zygmund class by Matrix - Cesaro summability means of Fourier series has been established.

Keywords : Degree of approximation , Weighted Zygmund class, Matrix – Cesaro mean, Cesaro means, Fourier series.

MSC : 41A24 , 41A25 , 42B05 , 42B08

1. Introduction

The degree of approximation of function belonging to different classes like Lip α, (Lip α, p), Lip(ξ(t),p) , Lip(Lp , ξ(t)) have been studied by many mathematician using different summability means. The error estimation of function in Lipschitz and Zygmund class using different means of Fourier series ans conjugate Fourier series have been great interest among the researcher.The generalized Zygmund class was introduced by Leindler [3] Moricz [5], moricz and Nemeth [6]etc. Recently Singh et. al. [9] Mishra et. al. [7], Kim [2] ,Shyamlal & Shireen [4] , Das et. Al. [1] find the results in Zygmund class by using different summability Means . In this paper we find the degree of approximation of function in the weighted Zygmund class by Matrix – Cesaro mean of Fourier series.

2. Definition

Let f be a periodic function of period 2π integrable in the sense of Lebesgue over

[π, - π]. Then the Fourier series of f given by

$$f(t) \approx \frac{a_o}{2} + \sum_{n=1}^{\infty}(a_n cos\, nx + b_n sin\, nx) \qquad ....(2.1)$$

Zygmund class z is defined as

$$Z = \{f \in C[-\pi,\pi] \,|f(x+t)+f(x-t)-2f(x)| = O(|t|)\}.$$

In this paper , we introduce a generalized Zygmund $Z^w(\alpha,\gamma)$ defined as

$$Z^{w}(\alpha,\gamma)=\left\{f\in C[-\pi,\pi]\left(\int_{-\pi}^{\pi}|f(x+t)+f(x-t)-2f(x)|^{\gamma}dx\right)^{\frac{1}{\gamma}}=O(|t|^{\alpha}\omega(t))\right\}$$

..........(2.2)

Where $\alpha \geq 0$, $\gamma \geq 1$ and ω is a continuous non negative and non decreasing

function. If we take $\alpha = 1, \omega = constant\ and\ \gamma \to \infty$, then $Z^{w}(\alpha,\gamma)$ class reduces to

the z class.

Now we define the weighted class as

$$W\left(Z_r^{(w)}\right)=\left\{f\in W\left(Z_r^{(w)}\right):1\le r\le\infty \sup_{t\neq 0}\frac{\|f(.+t)+f(.-t)-2f(.)sin^{\beta}(.)\|_r}{\omega(t)}\le\infty\right\}$$

where $\|f\|_r^{(w)} = \|f\|_r + \sup_{t\neq 0}\frac{\|f(.+t)+f(.-t)-2f(.)sin^{\beta}(.)\|_r}{\omega(t)}$ ....(2.3)

Let $T=(a_{n,k})$ be an infinite lower triangular matrix satisfying the conditions of regularity, i.e. $\sum_{k=0}^{\infty}|a_{n,k}| \leq M$, a finite constant .

Matrix – Cesaro means T(C1) of the sequence $\{S_n\}$ is given by

$t_n = \sum_{k=0}^{\infty} a_{n,n-k}\,\sigma_{n-k} = \sum_{k=0}^{\infty} a_{n,n-k}\frac{1}{n-k+1}\sum_{r=0}^{n-k} S_r$ . ...(2.4)

If $t_n \to S$ as n → ∞ , then the sequence $\{S_n\}$ or the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Matrix Cesaro means T(C1) method to S.

Important particular cases of Matrix -Cesaro means are :

(i) (N,Pn)C1 means ,when $a_{n,n-k} = \frac{P_k}{P_n}$, where $P_n = \sum_{k=0}^{\infty} P_k \neq 0$

(ii) (N,Pn)C1 means ,when $a_{n,n-k} = \frac{P_{n-k}}{P_n}$

(iii) (N,p,q) C1 means ,when $a_{n,n-k} = \frac{P_k q_{n-k}}{R_n}$, where $R_n = \sum_{k=0}^{\infty} P_k q_{n-k} \neq 0$

We write through the paper

$$\emptyset(x,t)=f(x+t)+f(x-t)-f(x)$$

$$K_n^{\Delta C}(t)=\frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{a_{n,n-k}}{(n-k+1)}\frac{sin^2(n-k+1)\frac{t}{2}}{sin^2\frac{t}{2}}$$

3.Main Result

In this paper we prove the following theorem.

Theorem –Let the lower triangular matrix $T = (a_{n,k})$ satisfying the following condition $a_{n,k} \geq 0 \ (n = 0, 1, 2, 3, \dots\dots; \ k = 0, 1, 2, \dots\dots. n)$

$\sum_{k=0}^{n} a_{n,k} = 1 \quad \sum_{k=0}^{n} |\Delta a_{n,k}| = o\left(\frac{1}{n+1}\right) \quad and \quad (n+1)a_{n,n} = o(1)$ ....(3.1)

Let f be a 2π periodic function ,Lebesgue integrable in $[0, 2\pi]$ and belonging to weighted Zygmund class W($Z_r^{(w)}$) $(r \geq 1)$. Then the degree of approximation of function f by matrix – Cesaro mean of Fourier series is given by

$$E_n(f) = inf \left\|t_n^{\Delta C} - f\right\|_r^v = o\left(\frac{1}{(n+1)} \int_{\frac{1}{n+1}}^{\pi} \frac{t^{\beta-2} \, w(t)}{v(t)} \, dt\right)$$

where $\omega(t)$ *and* $v(t)$ are the Zygmund moduli of continuity and $\frac{w(t)}{v(t)}$ is positive and non - decreasing .

4.Lemma –To prove the theorem we need the following lemma.

Lemma 4 (a) - For $0 \leq t \leq \frac{1}{n+1}$ we have

$$\left|K_n^{\Delta C}(t)\right| = O(n+1) \quad \text{.......(4.1)}$$

Proof - For $0 \leq t \leq \frac{1}{n+1}$ and $sin\frac{t}{2} \geq \frac{t}{\pi}$, $\sin nt \leq nt$, $|\cos t| \leq 1$ we have

:- $$K_n^{\Delta C}(t) = \frac{1}{2\pi} \sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} \frac{sin^2(n-k+1)\frac{t}{2}}{sin^2\frac{t}{2}}$$

$$= \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \, (n-k+1)$$

$$\leq \frac{n+1}{2\pi} \sum_{k=0}^{n} a_{n,n-k}$$

$$= \frac{n+1}{2\pi}$$

$$= O(n+1)$$

Lemma 4 (b) - For $\frac{1}{n+1} < t < \pi$, we have

$$|K_n^{\Delta C}(t)| = O\left(\frac{1}{(n+1)t^2}\right) \quad \text{.......(4.2)}$$

Proof - For $\frac{1}{n+1} < t < \pi, \quad sin\frac{t}{2} \geq \frac{t}{\pi}$

$$K_n^{\Delta C}(t) = \frac{1}{2\pi}\sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} \frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}$$

$$\leq \frac{1}{2\pi} \sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1} \frac{\pi^2}{t^2}$$, by Jordan's lemma

$$= \frac{\pi}{2t^2} \sum_{k=0}^{n} \frac{a_{n,n-k}}{n-k+1}$$

$$= \frac{\pi}{2t^2} O\left(\frac{1}{n+1}\right)$$

$$= O\left(\frac{1}{(n+1)t^2}\right)$$

Lemma 4(c) – Let $f \in Z_r^{(w)}$ then for $0 < t \leq \pi$

(i) $\|\phi(.,t)\|_p = O(w(t))$

(ii) $\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_r = \begin{cases} O(w(t) \\ O(w(y) \end{cases}$

(iii) If $\omega(t)$ and $v(t)$ are defined as in theorem then

$$\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_r = \left\{v(y)\frac{\omega(t)}{v(t)}\right.$$

Where $\phi(x,t) = f(x+t) + f(x-t) - 2f(x)$.

Lemma 4(d) - $\left\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\sin^\beta(.)\right\|_r = O\left(t^\beta\ v(y)\ \frac{\omega(t)}{v(t)}\right)$

Proof - Following Lemma 4(c) $|\sin^\beta t| \leq t^\beta$ and for v is positive non-decreasing $t \leq y$ we obtain

$$\left\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\sin^\beta(.)\right\|_r = O\left(t^\beta\ \omega(t)\right)$$

$$= O\left(t^\beta\ v(t)\ \frac{\omega(t)}{v(t)}\right)$$

$$\leq O\left(t^\beta\ v(t)\ \frac{\omega(t)}{v(t)}\right)$$

since $\frac{\omega(t)}{v(t)}$ is positive, non-decreasing if $t \geq y$ then $\frac{\omega(t)}{v(t)} \geq \frac{\omega(y)}{v(y)}$ so that

$$\left\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\ \sin^\beta(.)\right\|_r = O\left(t^\beta\ \omega(y)\right)$$

$$= O\left(t^\beta\ v(y)\ \frac{\omega(t)}{v(t)}\right)$$

5.Proof of Theorem 3

Let $s_n(f:x)$ denotes the nth partial sum of Fourier series given in (2.1) then we have

$$s_n(f:x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \emptyset(x,t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt \qquad ....(5.1)$$

The (C, 1) transform i.e. σ_n of s_n is given by

$$\frac{1}{n+1}\sum_{k=0}^{n}(s_k(x) - f(x)) = \frac{1}{2(n+1)\pi}\int_0^{\pi}\frac{\emptyset(t)}{\sin\frac{t}{2}} \sum_{k=0}^{n} \sin\left(k+\frac{1}{2}\right)t\, dt$$

$$\sigma_n(x) - f(x) = \frac{1}{2(n+1)\pi}\int_0^{\pi}\emptyset(t)\frac{\sin^2(n+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\, dt$$

The matrix means of the sequence $\{\sigma_n\}$ is given by

$$\sum_{k=0}^{n} a_{n,k}\big(\sigma_k(x) - f(x)\big) = \int_0^{\pi}\emptyset(t)\frac{1}{2\pi}\sum_{k=0}^{n}\frac{1}{(k+1)}\frac{\sin^2(k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}dt$$

$$\sum_{k=0}^{n} a_{n,k}\big(\sigma_{n-k}(x) - f(x)\big) = \int_0^{\pi}\emptyset(t)\frac{1}{2\pi}\sum_{k=0}^{n}\frac{1}{(n-k+1)}\frac{\sin^2(n-k+1)\frac{t}{2}}{\sin^2\frac{t}{2}}dt$$

$$t_n(x) - f(x) = \int_0^{\pi}\emptyset(t)K_n^{\Delta C}(t)d\,t \qquad ....(5.2)$$

Let $l_n(x) = t_n^{\Delta E}(x) - f(x) = \int_0^{\pi}\emptyset(x,t)\, K_n^{\Delta C}\ (t)\, dt$ then

$$l_n(x+y) + l_n(x-y) - 2\, l_n(x) = \int_0^{\pi}[\phi(x+y,t) + \phi(x-y,t) - 2\phi(x,t)]\, K_n^{\Delta C}(t)\, dt$$

Now $\big(l_n(.+y) + l_n(.-y) - 2\, l_n(.)\, \sin^{\beta}(.)\big)$

$$= \int_0^{\pi}[\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\sin^{\beta}(.)]\, K_n^{\Delta C}(t)\, dt$$

$$\|l_n(.+y) + l_n(.-y) - 2\, l_n(.)\sin^{\beta}(.)\|_r = \int_0^{\pi}\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\sin^{\beta}(.)\|_r K_n^{\Delta C}(t)\ dt$$

$$= \int_0^{\frac{1}{n+1}}\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\sin^{\beta}(.)\|_r K_n^{\Delta C}(t)\, dt$$

$$+ \int_{\frac{1}{n+1}}^{\pi}\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\sin^{\beta}(.)\|_r\ K_n^{\Delta C}(t)\, dt$$

$$= I_1 + I_2\ (say) \qquad ....(5.3)$$

The function Let $f \in W(Z_r^{(w)})$ implies $\emptyset \in W(Z_r^{(w)})$.

Using Lemma 4(a) and 4(d) and the monotonically of $\frac{\omega(t)}{v(t)}$ with respect to t we have

$$I_1 = \int_0^{\frac{1}{n+1}} \left\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)sin^\beta\right\|_r \; K_n^{\Delta E}(t)\; dt$$

$$= O\left(\int_0^{\frac{1}{n+1}} \left(v(y)\,\frac{t^\beta \omega(t)}{v(t)}\right)(n+1)\,dt\right)$$

$$= O\left((n+1)\;\; v(y)\int_0^{\frac{1}{n+1}} \frac{t^\beta \omega(t)}{v(t)}\; dt\right)$$

$$= O\left((n+1)\;\; v(y)\;\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\int_0^{\frac{1}{n+1}} t^\beta\; dt\right)$$

$$= O\left((n+1)^{-\beta}\;\; v(y)\;\;\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right) \qquad ....(5.4)$$

Using Lemma 4(b) and 4(d) we have

$$I_2 = \int_{\frac{1}{n+1}}^{\pi} \left\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)sin^\beta(.)\right\|_p \; K_n^{\Delta E}(t)\,dt$$

$$= O\left(\int_{\frac{1}{n+1}}^{\pi} \left(v(y)\frac{t^\beta \omega(t)}{v(t)}\right)(n+1)^{-1}t^{-2}\,dt\right)$$

$$= O\left((n+1)^{-1}v(y)\;\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{\beta-2}\omega(t)}{v(t)}\right)\; dt\right)$$

..............(5.5)

From (5.3) (5.4) and (5.5) we get

$\left\|l_n(.+y) + l_n(.-y) - 2l_n(.)\;\; sin^\beta(.)\right\|_r = \; O\left((n+1)^{-\beta}\; v(y)\;\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right) + \;\; O\Big((n+$
$1)^{-1}v(y)\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{\beta-2}\omega(t)}{v(t)}\right)\;\; dt\Big).$

$$\begin{matrix} sup \\ y \neq 0 \end{matrix} \frac{\|l_n(.+y)+l_n(.-y)-2l_n(.)sin^{\beta}(.)\|_r}{v(y)} = O\left((n+1)^{-\beta}\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right) + o\left((n+1)^{-1}\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{\beta-2}\omega(t)}{v(t)}\right)\right) \quad ....(5.6)$$

Clearly $\emptyset(x,t) = |f(x+t)+f(x-t)-2f(x)|$

Now applying Minkowski's inequality we have

$$\|\emptyset(x;t)\|_r = \|f(x+t)+f(x-t)-2f(x)\|_r$$

Again using Lemma we have

$$\left\|l_n(.)sin^{\beta}(.)\right\|_r \leq \left(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}.\right)\left\|\phi(.,t)\, sin^{\beta}(.)\right\|\left|\ K_n^{\Delta E}(t)\right|\,dt$$

$$= O\left((n+1)\int_0^{\frac{1}{n+1}} t^{\beta}\omega(t)dt\right) + O\left((n+1)^{-1}\int_{\frac{1}{n+1}}^{\pi} t^{\beta-2}\omega(t)\,dt\right)$$

$$= O\left((n+1)\omega\left(\frac{1}{n+1}\right)\int_0^{\frac{1}{n+1}} t^{\beta}dt + \left((n+1)^{-1}\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t^{2-\beta}}\,dt\right)\right)$$

$$= O\left((n+1)^{-\beta}\,\omega\left(\frac{1}{n+1}\right)\right) + o\left((n+1)^{-1}\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t^{2-\beta}}\,dt\right) \quad ...\ (5.7)$$

From (6.7) and (6.8) we have

$$\left\|l_n(.)sin^{\beta}(.)\right\|_r^{v} = \left\|l_n(.)sin^{\beta}(.)\right\|_r + \begin{matrix} sup \\ y \neq 0 \end{matrix}\frac{\|l_n(.+y)+l_n(.-y)-2l_n(.)sin^{\beta}(.)\|_r}{v(y)}$$

$$= o\left((n+1)^{-\beta}\,\omega\left(\frac{1}{n+1}\right)\right) + o\left((n+1)^{-1}\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t^{2-\beta}}\,dt\right) +$$

$$o\left((n+1)^{-\beta}\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right) + o\left((n+1)^{-1}\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{\beta-2}\omega(t)}{v(t)}\right)\right)$$

$$= \sum_{i=1}^{4} J_i \quad ...\ (5.8)$$

Now we write J_1 in terms of J_3 *and* J_2 , J_3 *in* term of J_4.

In view of the monotonicity of v(t) we have

$$\omega(t) = \left(\frac{\omega(t)}{v(t)}\right)\ v(t) \leq\ v(\pi)\left(\frac{\omega(t)}{v(t)}\right)\ v(t) = o\left(\frac{\omega(t)}{v(t)}\right) \quad for\ 0 < t \leq \pi$$

Therefore we can write

$$J_1 = O(J_3)$$

Again using monotonicity of v(t)

$$J_2 = (n+1)^{-1} \int_{\frac{1}{n+1}}^{\pi} t^{\beta-2} \frac{\omega(t)}{v(t)} v(t) dt \leq (n+1)^{-1} v(\pi) \int_{\frac{1}{n+1}}^{\pi} \left(t^{\beta-2} \frac{\omega(t)}{v(t)} \right) dt$$

$$\leq (n+1)^{-1} \int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{\beta-2}\omega(t)}{v(t)} \right) dt = O(J_4)$$

.... (5.9)

Using $\frac{\omega(t)}{v(t)}$ is positive and non- decreasing, we have

$$J_4 = (n+1)^{-1} \int_{\frac{1}{n+1}}^{\pi} \left(t^{\beta-2} \frac{\omega(t)}{v(t)} \right) dt$$

$$\geq (n+1)^{-1} \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)} \int_{\frac{1}{n+1}}^{\pi} t^{\beta-2} dt \geq (n+1)^{-1} \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)} \frac{1}{(n+1)^{\beta-1}}$$

$$\geq (n+1)^{-\beta} \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}$$

Therefore we can write

$J_3 = O(J_4)$ .... (5.10)

so we have

$$\left\| l_n(.) sin^{\beta}(.) \right\|_r^v = O(J_4) = O\left((n+1)^{-1} \int_{\frac{1}{n+1}}^{\pi} t^{\beta-2} \frac{\omega(t)}{v(t)} \right) dt \Bigg)$$

Hence

$$E_n(f) = \inf \left\| l_n(.) sin^{\beta}(.) \right\|_r^v = O\left((n+1)^{-1} \int_{\frac{1}{n+1}}^{\pi} t^{\beta-2} \left(\frac{\omega(t)}{v(t)} \right) dt \right)$$

This complete the proof.

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