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COMPLETELY HOMOGENEOUS GENERALIZED TOPOLOGICAL SPACES

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ABSTRACT

In this paper we try to characterize completely homogeneous generalized topologies and here we prove results without loss of generality for completely homogeneous strong generalized topologies only. If μ is a generalized topology on X which is not strong, then the results we prove here still hold if we replace X by M_μ.

Keywords *: Topological spaces, homogeneous, cyclic and infinite etc.*

Introduction :

Homogeneity in topological spaces is studied by many mathematicians. John Ginsburg in his paper [1] proved a simple representation theorem for finite topological spaces which are homogeneous. In this paper we characterize completely homogeneous generalized topological spaces. In the following papers we deal with homogeneous generalized topological spaces in a cyclic ordered set. We try to find out new homogeneous generalized topological spaces by considering the join of homogeneous generalized topologies and discuss the properties.

Let X be a nonempty set and μ be a generalized topology on X. We de note the union of all open sets in (X, μ) by M_μ. Let us recall the definition of homogeneous generalized topological space.

We use some set theoretic results throughout this paper. Consider a nonempty set X and A and B are subsets of X. Then there exists a bijection on X, which maps A onto B if and only if $|A| = |B|$ and $|X \setminus A| = |X \setminus B|$. If X is an infinite set, it is possible to choose subsets A and B of X such that $A \cup B = X$, $A \cap B = \emptyset$, and $|A| = |X| = |B|$ since $\alpha + \alpha = \alpha$ for any infinite cardinal α [2].

Throughout this chapter X will denote a nonempty ordinary set unless otherwise stated.

Examples of completely homogeneous generalized topologies :

1. $\{\emptyset\}$, $\{\emptyset, X\}$ and $P(X)$ on any set X.
2. $\mu = \{\emptyset, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}$ on $X = \{a, b, c\}$.
3. $\tau = \{G \subseteq X : G \text{ is infinite}\} \cup \{\emptyset\}$ is a completely homogeneous generalized topology on an infinite set X.

Lemma 1.1. *Let (X, μ) be a completely homogeneous generalized topological space and C be a subset of X such that $|C| < |X|$. If C is open in (X, μ), then every subset B of X such that $|B| = |C|$ is also open in (X, μ).*

Proof. Let $B \subseteq X$ and $|B| = |C|$. Since $|C| < |X|$, we have $|X \setminus C| = |X \setminus B|$. Then there exists a bijection f on X, which map C onto B, consequently f is an open map since every bijection is a homeomorphism in a completely homogeneous generalized topological space and hence $f(C) = B$ is open in (X, μ).

Lemma 1.2. *Let (X, μ) be a completely homogeneous generalized topological space and let $C \subseteq X$, $C = \emptyset$, is open in (X, μ). Then supersets of C are also open in (X, μ).*

Proof. Let $C \emptyset D \subseteq X$, then there exists an element $y \in D$ and $y \,/\!\in C$. Let $x \notin C$. Consider the bijection f on X which map x onto y and y onto x and f is the identity map on all other elements. But every bijection is a homeomorphism on X and hence f is a homeomorphism on X. Since f is an open map, $f(C) = (C \setminus \{x\}) \cup \{y\}$ is open in (X, μ). Then $C \cup \{y\}$ is open since it is the union of two open sets, $C \cup \{y\} = C \cup (C \setminus \{x\} \cup \{y\})$. Thus D is open since D can be written as $D = \bigcup_{\substack{y \in X \\ y \notin C}} \left(C \cup \{y\} \right)$. Hence the result.

Clearly the converse of previous lemma is not true. For example consider the generalized topology $\mu = \{\emptyset, \{a, b\}, \{b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c\}, X\}$ on the set $X = \{a, b, c, d\}$. It can be easily verified that the supersets of nonempty open sets are again open in (X, μ), but is not completely homogeneous generalized topological space [5].

Larson determined the completely homogeneous topologies in his paper [3]. He proved the following theorem.

Theorem 1.3. *[3] The only completely homogeneous topologies on a set X are:*

1. *The indiscrete topology*
2. *The discrete topology*
3. *Topologies of the form* $\{G \subseteq X : |X \setminus G| < m\} \cup \{\emptyset\}$, *where* $\aleph_0 \leq m \leq |X|$.

Next is a characterization theorem for completely homogeneous generalized topological spaces with a nonempty open subset of cardinality strictly less than that of X.

Theorem 1.4. *Let (X, μ) be a generalized topological space and C be a nonempty open subset of X such that $|C| < |X|$. Then μ is completely homogeneous generalized topology if and only if* $\mu = \{G \subseteq X : |G| \geq m\} \cup \{\emptyset\}$ *where* $m < |X|$.

Proof. Assume (X, μ) is completely homogeneous. If (X, μ) is a topological space, then we may use the preceding theorem by Larson. We observe that the only completely homogeneous topologies on a finite set are indiscrete and discrete topologies and if X is infinite, then μ is either discrete or every nonempty open set has cardinality the same as that of X. Therefore, since μ contain C and by Theorem 1.3, μ is completely homogeneous if and only if μ is $P(X) = \{G \subseteq X : |G| \geq 1\} \cup \{\emptyset\}$, if (X, μ) is a topological space.

Let (X, μ) be a completely homogeneous generalized topological space and not a topological space. Now consider the set $S = \{|G| : \emptyset = G \in \mu \text{ and } |G| < |X|\}$. The set S is nonempty since $|C| \in S$. Let m be the smallest element in S. Then there exists a set $D\emptyset\ X$ such that $|D| = m < |X|$ and D is open in (X, μ). By Lemma 1.1, if $B \subseteq X$ and $|B| = |D|$, then B is also open in (X, μ). Also by Lemma 1.2, supersets of B is also open for every $B \subseteq X$ such that $|B| = |D|$. On the other hand, nonempty subsets of cardinality less than m are not open. Thus μ is of the form $\{G \subseteq X : |G| \geq m\} \cup \{\emptyset\}$, where $m < |X|$. Conversely, if $\mu = \{G \subseteq X : |G| \geq m\} \cup \{\emptyset\}$ for some $m < |X|$, then it can be easily verified that μ is a completely homogeneous generalized topology on X.

Now consider the generalized topological space in which every non empty open set has cardinality same as that of whole set. Next we enquire when does this generalized topology completely homogeneous. First we prove some Lemmas.

Lemma 1.5. *Let μ be a completely homogeneous generalized topology on an infinite set X. Let G be an open subset of X with $|G| = |X|$ and $|G^c| = |X|$. Then every $H \subseteq X$ such that $|H| = |G|$ is open in (X, μ).*

Proof. Let $H \subseteq X$ and $|H| = |G|$. Since H is an infinite set, there exist disjoint subsets $A, B \subseteq H$ such that $|A| = |B| = |H|$ and $A \cup B = H$. Then $B \subseteq A^c$ and $|H| = |B| \leq |A^c| \leq |X| = |H|$. Hence $|A^c| = |H|$. But $|H| = |G| = |X| = |G^c|$ getting $|A^c| = |G^c|$. Also $|A| = |H| = |G|$ getting $|A| = |G|$. Then there exists a bijection f on X, which map A onto G. Since (X, μ) is a completely homogeneous generalized topological space, f is a homeomorphism. Consequently A is an open set since $A = f^{-1}(G)$ and G is open. But H is a superset of A. Hence by Lemma 1.2, H is open in (X, μ).

Lemma 1.6. *Let μ be a completely homogeneous generalized topology on an infinite set X. Let G be a subset of X with $|G| = |X|$ and $|G^c| < |X|$. If G is open in (X, μ), then for every $H \subseteq X$ such that $|H| = |G|$ and $|H^c| \leq |G^c|$ are open in (X, μ).*

Proof. Let H be a subset of X such that $|H| = |G|$ and $|H^c| \leq |G^c|$. If $|H^c| = |G^c|$, then there exists a bijection, say f, on X mapping G onto H. Since every bijection is a homeomorphism, $f(G) = H$ is open in (X, μ).

Now assume $|H^c| < |G^c|$. Consider a subset $A \subseteq H$ such that $|A \cup H^c| = |G^c|$. But $A \cup H^c = (H \setminus A)^c$. Therefore $|(H \setminus A)^c| = |G^c|$.

Case 1: G^c is a finite set.

Then H^c is finite and consequently A has to be finite and since $|H \setminus A|+|A| = |H|$, we have $|H \setminus A| = |H| = |G|$. Thus we obtain $|H \setminus A| = |G|$ and $|(H \setminus A)^c| = |G^c|$. Then there exists a bijection on X mapping G onto $H \setminus A$ and by proceeding as earlier we get $H \setminus A$ is open in (X, μ). But $H \setminus A \subseteq H$, therefore by Lemma 1.2, H is also open in (X, μ).

Case 2: G^c is an infinite set.

Note that $|(H \setminus A)^c| = |G^c|$, i.e., $|H^c \cup A| = |G^c|$ implying $|H^c| + |A| = |G^c|$. Since $|G^c|$ is infinite and $|H^c| < |G^c|$, we have $|A| = |G^c|$. But $|G^c| < |X| = |H|$, resulting $|A| < |H|$. Consider $|H \setminus A| + |A| = |H|$, consequently $|H \setminus A| = |H|$ since $|H|$ is infinite. Thus we have $|H \setminus A| = |G|$ and $|(H \setminus A)^c| = |G^c|$ and by similar arguments as in Case 1, we can prove that H is an open subset of X. Hence the proof is complete.

The previous lemmas enable us to prove the following characterization theorem.

The following definition is adopted from [4].

Definition 1.7. *The successor of a cardinal m is the least cardinal greater than m. A cardinal is said to be a limit cardinal if it is not the successor of a cardinal.*

Theorem 1.8. *Let μ be a generalized topology on an infinite set X and every $\emptyset \neq G \in \mu$ has cardinality as that of X. Then μ is a completely homogeneous generalized topology if and only if μ is of one of the following form.*

1. $\{G \subseteq X : |G| = |X|\} \cup \{\emptyset\}$.

2. $\{G \subseteq X : |G^c| \leq m\} \cup \{\emptyset\}$, *where* $m < |X|$.

3. $\{G \subseteq X : |G^c| < m\} \cup \{\emptyset\}$, *where* $m \leq |X|$ *and m is a limit cardinal,* $m \neq 0$.

Proof. Let (X, μ) be a completely homogeneous generalized topological space in which every $\emptyset \neq G \in \mu$ has $|G| = |X|$.

By Lemma 1.5, if for some $G \in \mu$ has $|G^c| = |X|$, then every $H \subseteq X$ such that $|H| = |G|$ is open in (X, μ). In other words, $\{H \subseteq X : |H| = |X|\} \subseteq \mu$. Moreover by the assumption every nonempty open set has cardinality the same as that of X. Therefore $\mu = \{G \subseteq X : |G| = |X|\} \cup \{\emptyset\}$.

Now suppose for every $\emptyset \neq G \in \mu$, $|G| = |X|$ and $|G^c| < |X|$. Consider the set $F = \{|G^c| : G \in \mu, G \neq \emptyset\}$. Since F is bounded by $|X|$, supremum of F exists and let $m = \sup F$.

Case 1: There exists $\emptyset \neq K \in \mu$ such that $|K^c| = m$

Then for every $\emptyset \neq G \in \mu$, $|G^c| \leq m$, i.e., $\mu \subseteq \{G \subseteq X : |G^c| \leq m\} \cup \{\emptyset\}$. Now by Lemma 1.6, every $G \subseteq X$ such that $|G^c| \leq |K^c|$, is also open in (X, μ). Hence $\{G \subseteq X : |G^c| \leq m\} \subseteq \mu$. Also note that here $m = |X|$. Hence $\mu = \{G \subseteq X : |G^c| \leq m\} \cup \{\emptyset\}$, where $m < |X|$.

Case 2: For every open set $\emptyset \neq G \in \mu$, $|G^c| = m$ and $m \neq 0$.

For every $\emptyset \neq G \in \mu$, $|G^c| < m$, i.e., $\mu \subseteq \{G \subseteq X : |G^c| < m\} \cup \{\emptyset\}$. Now since $m = supF$, given any $\alpha < m$, there exists $H \in \mu$ such that $|H^c| = \alpha$. Then every set $M \subseteq X$, with $|M^c| = \alpha$, is open in (X, μ). Moreover by Lemma 1.6, every set $U \subseteq X$ with $|U^c| < \alpha$ is also open in (X, μ). This is true for every cardinal number $\alpha < m$. Hence $\{G \subseteq X : |G^c| < m\} \cup \{\emptyset\} \subseteq \mu$ and thus we get $\mu = \{G \subseteq X : |G^c| < m\} \cup \{\emptyset\}$, where $m \leq |X|$.

If m is not a limit cardinal then there exists a cardinal n such that m is the successor of n. Therefore μ can be written as $\mu = \{G \subseteq X : |G^c| \leq n\} \cup \{\emptyset\}$. Hence if m is a limit cardinal and $m \neq 0$, then μ takes the form $\{G \subseteq X : |G^c| < m\} \cup \{\emptyset\}$.

Now the converse part of the theorem, we can easily verify that the generalized topologies listed in the theorem are completely homogeneous. Hence the proof.

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