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COMMON FIXED-POINT THEOREM FOR FUZZY MAPPINGS IN COMPLETE PARTIAL METRIC SPACES

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ABSTRACT: -In this paper we obtain some common fixed-point theorems for fuzzy mappings in complete partial metric-spaces.

KEY WORDS: -Fuzzy Mapping, Fixed Points, Partial Metric-Spaces, complete partial metric-spaces.

1 INTRODUCTION: -In 1994, Matthews [35] introduced the notion of a partial metric space and obtained, among other results, a Banach contraction mapping for these spaces, later on, O'Neil [40] generalized Matthews's notation of partial metric, in order to establish connections between these structures and the topological aspects of domain theory. Here, we obtained a fixed-point theorem for complete partial metric space in the sense of O'Neill [40]. Thus, Matthews' fixed-point theorem follows as special case of our result.

Throughout this paper the letters R, and N will denote the set of real numbers, and the set of natural numbers, respectively.

The notion of a partial metric space was introduced by Matthews in [35] as a part of the study of denotational semantics of dataflow networks. In particular, he established the precise relationship between partial metric spaces and the so-called weighable quasi-metric spaces and proved a partial metric generalization of Banach's contraction mapping theorem.

Let us recall that a partial metric on a (nonempty) set X is a function $p: X \times X \to R^{+}$ such that for all x, y, z ∈ X

I. $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;

II. $p(x,x) \leq p(x,y)$;

III. $p(x,y) = p(y,x)$;

IV. $p(x,z) \leq px,y) + p(y,z) - p(y,y)$

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

In [40], O' Neill proposed one significant change to Matthews's definition of the partial metrics, and that was to extend range from R^{+} to R.

In the following, partial metrics in the O'Neill sense will be called dualistic partial metrics and a pair (X, p) such that X is a nonempty set and p is a dualistic partial on X will be called a dualistic partial metric space.

In this way, O'Neill developed several connections between partial metrics and those topological aspects of domain theory. Moreover, the pair (R, p), where p (x, y) = x v y for all x, y ∈ R, provides a paradigmatic example of a dualistic partial metric (or partial) metric spaces which are interesting from a computational point of view may be found in [13], [35] ,[52] , [55], etc.

Each dualistic partial metric p on X generates a topology $\tau_{(p)}$ on X, which has as a base the family of open p-balls $\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$, Where

$B_p(x,\epsilon) = \{y \in X: p(x,y) < p(x,x) + \epsilon\}$,, for all $x \in X$ and $\in > 0$.

From this fact it immediately follows that a sequence $\{x_n\}_{n\in N}$ in a dualistic partial metric space (X, p) converges to a point x ∈ X if and only if $p(x, x) = lim_{n\to\infty} p(x, x_n)$

According to [40] (compare [35], a sequence $\{x_n\}_{n\in N}$ in a dualistic partial metric space (X, p) is called a Cauchy sequence if there exists $\lim_{n,m\to\infty} p(x_n, x_m)$.

A dualistic partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in N}$ in X convergence to $\tau(p)$ to a point x ∈ X such that, $p(x,x) = \lim_{n\to\infty} p(x_n, x_m)$

We indicated above, and motivated by applications in program verification, Matthews obtained in [35], a Banach fixed point theorem for complete partial metric spaces. Since (complete) dualistic partial metrics provide a new approach to generalizing both the domain theoretic and the metric approach to semantics (see [40], p 314), it seems interesting to obtain a Banach fixed point theorem in the realm of dualistic partial metric spaces. In this paper we present theorems of this type. In particular, Matthews's contraction mapping theorem will be deduced as a special case of our result.

2 PRELIMINARIES: -

Before stating our main result, we establish some (essentially known) correspondences between dualistic partial metric spaces and quasi-metric spaces.

Our basis references for quasi-metric spaces are [16] and [28]. In our context by a quasi-metric on a set X is a nonnegative real-valued function d on X × X such that for all x, y, z ∈ X:

I. $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$

II. $d(x, y) < d(x, z) + d(z, y)$

A quasi-metric space is a pair (X, d) such that X is a nonempty set and d is a quasi-metric on X.

Each quasi-metric d on X generates a τ_0 topology $\tau(d)$ on X which has a base the family of open d-balls {$\{B_d(x, \varepsilon): x \in X, \varepsilon > 0\}$, where

$B_d(x, \varepsilon) = \{y \in X: d(x, y) < \varepsilon\}$, for all x ∈ X and $\varepsilon > 0$.

If d is quasi-metric on X, then the function d' defined on X × X by

$d'(x, y) = \max\{d(x, y), d(y, x)\}$, is a metric on X.

Lemma.2.1: - If (X, p) is a dualistic partial metric space, then the function d_p: X' × X'→ R^+ defined by $d_p(x, y) = p(x, y) - p(x, x)$, is a quasi-metric on X such that T(p) = T(d_p) .

Proof: -Consider x, y ∈ X. Then $d_p(x, y) = p(x, y) - p(x, x)$ is always nonnegative because of $p(x, x) \leq p(x, y)$. Now, we have to check that d_p is actually a quasi-metric on X.

Let x, y, z ∈ X. It is obvious that x = y provides that $d_p(x, x) = d_p(y, x) = 0$. Moreover, if $d_p(x, y) = d_p(y, x) = 0$, then, $p(x, x) = p(y, x) = p(x, y) = 0$. Hence, we obtain that $x = y$, since $p(x, y) = p(x, x) = p(y, y)$. Furthermore

$$d_p(x, y) = p(x, y) - p(x, x)$$
$$\leq p(x, z) + p(z, y) - p(z, z) - p(x, x)$$

$=d_p(x,z)+d_p(z,y)$

Finally we show that T(d) = T(d_p) Indeed, let x ∈ X and $\varepsilon > 0$ *and* consider y ∈ $B_{d_p}(x,\varepsilon)$. Then $d_p(x, y) = p(x, y) — p(x, x) < \varepsilon$ and, hence $p(x,y) < \varepsilon + p(x,x)$

Consequently y ∈ B_p(x,ε) and T(d_p) ⊆ T(d)

Conversely if y∈ $B_p(\mathrm{x},\varepsilon)$ we have that $p(x,y) < \varepsilon + p(x,x)$

Thus $dp(x, y) = p(x, y) — p(x, x) < \varepsilon$, y ∈ Bp(x, ε) and T(d) ⊆ T(d_p)

Lemma 2.2: - A dualistic partial metric space (X. p) is complete if and only if the metric space (X, $(d_p)^s$) is complete. Furthermore

$\lim_{n\to\infty} (d_p)^*(a,x_n) = 0$ If and only if

$p(\mathrm{a, a}) = \lim_{n,\to\infty} p(\,a,x_n) = \lim_{n,m\,\to\infty} p(\,x_{n,},x_m)$

Proof: - First, we show that every Cauchy sequence in (X, p) a Cauchy sequence in $(\mathrm{X},\ (d_p)^s)$. To this end let $\{x_n\}_{n\in N}$ be a Cauchy sequence in (X, p), Then there exists a ∈R such that, given ∈ > 0, there is $n_\in$ ∈ N with $|p(x_n,x_m) - \alpha| < \in/2$ for all n, m ≥ $n_\in$

$d_p(x_n,x_m) = p(x_n,x_m) - p(x_n,x_n)$

$= |p(x_n,x_m) - \alpha + \alpha - p(x_n,x_n)|$

$\leq |p(x_n,x_m) - \alpha| + |\alpha - p(x_n,x_n)|$

$< \in/2 + \in/2 = \in$ 2.1

For all n, m ≥ $n_\in$. similarly we show $d\ (x_m x_n\ ) < \in$ for all n, m ≥ $n_\in$. We conclude that $\{x_n\}_{n\in N}$ is a Cauchy sequence in (X, $(d_p)^s$).

Next, we prove that completeness of (X,$(d_p)^s$) implies completeness of (X, p). Indeed, if $\{x_n\}_{n\in N}$ is a Cauchy sequence in (X, p) then it is also a Cauchy sequence in (X, $(d_p)^s$). Since the metric space (X, $(dp)^*$).is complete we deduce that there exists y∈X such that $\lim_{n\to\infty} (d_p)^s(y,\ x_n) = 0$. By (2.1) we follow that $\{x_n\}_{n\in N}$ is a convergent in sequence in (X, p). Next, we prove that $\lim_{n\to\infty} p(\,x_n,x_m) = p(y,y)$.

Since $\{x_n\}_{n\in N}$ is a Cauchy sequence in (X, p). It is sufficient to see that $\lim_{n\to\infty} p(\,x_n,x_n) = p(y,y)$. Let ∈ > 0 then there exists n_0 ∈ N such that $(d_p)^s(y,\ x_n) < \in/2$ whenever $n \geq\ n_0$

Thus

$$|p(y,y)-p(x_n,x_n)| \leq |p(y,y)-p(y,x_n)| + |p(y,x_n)-p(x_n,x_n)|$$

$$= d_p(y,x_n) + d_p(x_n,y)$$

$$< 2(d_p)^*(y,x_n) < \in$$

Whenever, $n \geq n_0$. This shows that (X, p) is complete.

Now we prove that every Cauchy sequence $\{x_n\}_{n\in N}$ in (X, $(d_p)^s$) is a Cauchy sequence in (X, p). Let $\in = 1/2$. Then there exists $n_0 \in N$ such that $d_p(x_n,x_m) < 1/2$ for all $n,m \geq n_0$. Since

$$d_p(x_n,x_{n_0}) + p(x_{n,}x_{n,}) = d_p(x_{n_0},x_n) + p(x_{n_0}\, x_{n_0})$$

Then

$$|p(x_n,x_n)| = |d_p(x_{n_0},x_n) + p(x_{n_0}\, x_{n_0}) - p(x_{n,}x_{n_0})|$$

$$\leq d_p(x_{n_0},x_n) + |p(x_{n_0}\, x_{n_0})| + d_p(x_n,x_{n_0})$$

$$\leq 2(d_p)^*(x_n,x_{n_0}) + |p(x_{n_0}\, x_{n_0})|$$

$$< 1 + |p(x_{n_0}\, x_{n_0})|$$

Consequently the sequence $\{p(x_n,x_n)\}$ is bounded in R, and so there exists z ∈R such that a subsequence $\{p(x_{n_k},x_{n_k})\}$ is convergent to z i.e. $\lim_{n\to\infty}\{p(x_{n_k},x_{n_k})\} =$ z.

It remains to prove that $\{p(x_n,x_n)\}$ is a Cauchy sequence in R. Since $\{x_n\}_{n\in N}$ is a Cauchy sequence in (X, $(d_p)^s$ given ∈ > 0 there existsnc n_0 ∈ N such that

$(d_p)^s(x_n,x_n) < \frac{\in}{2}$ for all $n,m \geq n_\in$. Thus for all $,m \geq n_\in$,

$$|p(x_n,x_n) - p(x_{m,}x_m)| = |d_p(x_m,x_n) - d_p(x_{n,}x_m)|$$

$$\leq 2(d_p)^s(x_m,x_n) < \in$$

Because of $p(x_n,x_n) = d_p(x_m,x_n) + p(x_m,x_m) - d_p(x_n,x_m)$

Therefore $\lim_{n\to\infty} p(x_n,x_n) = z$

On the other hand,

$$|p(x_n,x_m) - z| = |p(x_n,x_m) - p(x_n,x_n) + p(x_n,x_n) - z|$$

$\leq d_p(x_n,x_m) + |p(x_n,x_n) - z| < \in$, for all n, m $\geq n_\in$. Hence $\lim_{n,m\to\infty} p(x_n,x_m) =$ z and $\{x_n\}_{n\in N}$ is a Cauchy sequence in (x, p). We shall have established the lemma if we proof that (X,$(d_p)^s$) is complete if so is (X, p). Let $\{x_n\}_{n\in N}$ be a Cauchy sequence in (X,$(d_p)^s$). Then $\{x_n\}_{n\in N}$ is a Cauchy sequence in (X, p), and so it is convergent to a point y ∈ X with

$$\lim_{n,m\to\infty}(x_n, x_m) = \lim_{m\to\infty}(y, x_m) = p(y, y)$$

Then given $\in > 0$, there exists $n_\in \in N$, such that

$p(y, x_n) - p(y, y) < \in$ and $p(y, y) - p(x_n, x_m) < \in$, whenever $n \geq n_\in$

As consequence we have

$$d_p(y, x_n) = p(y, x_n) - p(y, y) < \in$$

And

$$d_p(x_n, y) = p(y, x_n) - p(x_n, x_n)$$

$\leq |p(y, x_n) - p(x_n, x_n)| + |p(y, y)p(x_n, x_n)| \leq 2\in$

Whenever $n \geq n_\in$.Therefore (X, $(d_p)^s$) is complete.

Finally, it is a simple matter to check that $\lim_{n\to\infty}(d_p)^s(z, x_n) = 0$ if and only if $p(z, z) = \lim_{n\to\infty} p(z, x_n) = \lim_{n,m\to\infty} p(x_n, x_m)$

3 MAIN THEOREMS: -

Theorem 3.l: - Let (X, p) be a complete dualistic partial metric space and F be a fuzzy mapping on X into W^* (X) such that there is a real number q with $0 \leq q < l$, satisfying

$$|D(F(x), F(y))| \leq q\left|\min\{p(x, y), D_\alpha(x, F(x)), D_\alpha(y, F(y))\}\right|$$

For all x, y ∈ X .Then F has a unique fixed point.

Proof: - Let $x_0 \in X$ be arbitrary and let $x_n = F(x_{n-1}) = F^n(x_0)$

$|p(F(x_0), F(x_0)| = |p(x_{1,}x_1)|$

$\leq |H(F(x_0), F(x_0)| \leq |D(F(x_0), F(x_0))| \leq q\left|\min\{p(x_0, x_0), D_\alpha(x_0, F(x_0)), D_\alpha(x_0, F(x_0))\}\right| = q\left|\min\{p(x_0, x_0), p(x_0, x_{1,}), p(x_0, x_{1,})\}\right| = q\,|p(x_0, x_0)|$

$|p(F(x_0), F(x_0)| \leq q|p(x_0, x_0)|$

And

$|p(F(x_1), F(x_1)| = |p(x_{2,}x_2)|$

$\leq |H(F(x_1), F(x_1)| \leq |D(F(x_1), F(x_1))| \leq q\left|\min\{p(x_1, x_1), D_\alpha(x_1, F(x_1)), D_\alpha(x_1, F(x_1))\}\right| = q\left|\min\{p(x_1, x_1), p(x_1, x_{2,}), p(x_1, x_{2,})\}\right| = q\,|p(x_1, x_1)| \leq q^2|p(x_0, x_0)|$

Similarly we can show that

$|p(F(x_n), F(x_n)| =\leq q^n\,|p(x_{0,}x_0)|$

Now

$|p(F(x_0), F(x_1)| = |p(x_{1,}x_2)|$

$\leq |H(F(x_0), F(x_1)| \leq |D(F(x_0), F(x_1))| \leq q|\min\{p(x_0, x_1), D_\alpha(x_0, F(x_0)), D_\alpha(x_1, F(x_1))\}| = q|\min\{p(x_0, x_{1,}), p(x_0, x_{1,}), p(x_1, x_{2,})\}|$

$= q\ |p(x_0, x_0)|$

$|p(F(x_0), F(x_1)| \leq \text{q}|p(x_0, x_1)|$

and

$|p(F(x_1), F(x_2)| = |p(x_2, x_2)|$

$\leq |H(F(x_1), F(x_2)| \leq |D(F(x_1), F(x_2))| \leq q|\min\{p(x_1, x_2), D_\alpha(x_1, F(x_1)), D_\alpha(x_2, F(x_2))\}| = q|\min\{p(x_1, x_2), p(x_1, x_2), p(x_2, x_{3,})\}|$

$= q\ |p(x_1, x_2)|$

$|p(F(x_1), F(x_2)| \leq \text{q}|p(x_1, x_2)|$

$\leq q^2|p(x_0, x_1)|$

Hence

$|p(F(x_n), F(x_{n+1})| \leq q^n\ |p(x_{0,}x_1)|$

Since by lemma 2.1

$$d_p(F(x_n), F(x_{n+1}) = p(F(x_n), F(x_{n+1})) - p(F(x_n), F(x_n))$$
$$d_p(F(x_n), F(x_{n+1}) + p(F(x_n), F(x_n)) = p(F(x_n), F(x_{n+1}))$$

We deduce that

$$d_p(F(x_n), F(x_{n+1}) + p(F(x_n), F(x_n)) \leq q^n|p(x_{0,}x_1)|$$

Hence

$$d_p(F(x_n), F(x_{n+1}) \leq q^n|p(x_{0,}x_1)| - p(F(x_n), F(x_n))$$
$$\leq q^n|p(x_{0,}x_1)| + q^n|p(x_{0,}x_0)|$$
$$= q^n\{|p(x_{0,}x_1)| + |p(x_{0,}x_0)|\}$$

Now let n, k $\in N$ then

$$d_p(F(x_n), F(x_{n+k})) \leq d_p(F(x_n), F(x_{n+1})) + d_p(F(x_{n+1}), F(x_{n+2}))$$
$$+ \cdots + \cdots$$
$$+ d_p(F(x_{n+k-1}), F(x_{n+k}))$$
$$\leq \{q^n + q^{n+1} + \cdots q^{n+k-1}\}\{|p(x_{0,}x_1)| + |p(x_{0,}x_0)|$$
$$< \frac{q^n}{1-q}\{|p(x_{0,}x_1)| + |p(x_{0,}x_0)|\}$$

Similarly we have

$$d_p\big(F(x_{n+1}),F(x_n)\big) < \frac{q^n}{1-q}\{\big|p\big(x_{0,}x_1\big)\big| + \big|p\big(x_{0,}x_0\big)\big|\}$$

Consequently $\{F(x_n)\}$ is a Cauchy sequence in the metric space $\big(X,(d_p)^*\big)$ which is complete. So there is z ∈ X such that $\lim_{n,m\to\infty}(d_p)^*(z,x_n) = 0$, we want to show that z is the unique fixed point of F. firstly, by lemma 2.2 we obtain

$$p(z,z) = \lim_{n\to\infty} p(z,(x_n)) = \lim_{n,m\to\infty} p(F(x_n),F(x_m))$$

Moreover, since

$$\lim_{n\to\infty} d_p(F(x_n),F(x_m)) = \lim_{n\to\infty} p(F(x_n),F(x_m))$$

We deduce from lemma 2.1 that

$$\lim_{n,m\to\infty} p(F(x_n),F(x_m)) = 0$$

Therefore,

$$p(z,z) = \lim_{n\to\infty} p(z,(x_n)) = 0$$

Now since,

$$|p(F(z),F(z))| \leq q|p(z,z)| = 0$$

It follows that

$$p\big(F(z),F(z)\big) = 0$$

On the other hand, since

$$|p(F(z),F(x_{n+1}))| \leq q|z,F(x_n))|$$

It follows that

$$\lim_{n\to\infty} p\,\big(\,z,F(x_n)\big) = 0$$

Thus F(z) is the limit point of $\{F(x_n)\}$ in $\big(X,\big(d_p\big)^*\big)$. Consequently,

$z \;= F(z).$

Finally let $z' \in X$ such that $F(z') = z'$ then

$|p(z,z')| = |p(F(z),F(\,z'))| \leq q|p(z,z')|$

$$\Rightarrow \quad z = z'.$$

Therefore z is a unique fixed point of F.

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