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ALGEBRAIC TECHNIQUES IN NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract: - When it comes to solving nonlinear differential equations, which are encountered in a wide variety of fields such as applied mathematics, engineering, and physics, algebraic approaches have developed into powerful tools. With the help of these methods, it is possible to find accurate answers in a methodical manner, which is something that is not always possible when relying on more conventional procedures. The Lie group approach, symmetry analysis, and the Hirota bilinear method are all examples of methods that reduce complex problems by transforming nonlinear equations into algebraic forms that can be solved. To be more specific, the Lie group approach is a method that seeks analytical solutions by simplifying differential equations through the investigation of symmetries that exist within them. Through the use of the Hirota method, nonlinear equations are transformed into bilinear form, which makes it simpler to construct solutions for many soliton problems. There are a number of algebraic approaches that can be utilized to determine integrability and to rank equations according to their solvability. One of these methods is the Painlevé analysis. When it comes to dealing with nonlinear phenomena such as wave propagation, fluid dynamics, or population dynamics, these approaches really shine. Algebraic methods are useful for generating numerical or approximate answers in situations where exact solutions are not attainable. These methods give light on the stability and qualitative behavior of solutions, which helps to generate solutions. Algebraic approaches have increased our ability to utilize nonlinear differential equations in a wide variety of engineering and scientific domains. This is because these methods provide a solid foundation for solving and comprehending these problems.

Keywords: - Riccati Equation, Abel Equatns, Cauchy-Kowalewski Theorem, Cauchy-Kowalski System, Unival Cauchy-Kowalski System.

I. INTRODUCTION

Throughout these notes, the primary focus is on systems of ordinary differential equations and the issues associated with their initial values. In this section, we will concentrate on nonlinear properties and phenomena, particularly those that make a substantial contribution to the physical phenomenon. Obtaining a solution to a differential equation may not be of great relevance if the solution in question does not exist in the physical model of the system or if it can only be obtained under extremely unusual circumstances. Stable equilibrium solutions are the only ones that actually materialize in real-world circumstances; these solutions represent configurations in which the physical system does not undergo any motion. This is the reason why stable equilibrium solutions are extremely rare. On account of the fact that even minor alterations to the system or its physical environment have the potential to rapidly throw it out of balance, an unstable equilibrium will not occur in the real world. In order to approximate the solution of nonlinear systems, numerical techniques are typically utilized. This is due to the fact that only a small percentage of these systems can be solved openly. The latter portion of this chapter will be devoted to discussing a number of fundamental techniques to initial value problems. Beginning with the straightforward Euler scheme, we will go on to the Runge-Kutta fourth order method, which is employed by a significant number of people. The category of stiff differential equations, which numerical analysts face a more significant threat from, is briefly discussed, and it is brought to our attention that numerical methods may not necessarily provide accurate findings. If you do not have a theoretical understanding of solution types, equilibrium points, and stability characteristics, you will not be able to determine whether or not numerical solutions, even those that come from quite popular and well-known packages, are reliable. When it comes to evaluating numerical schemes, it is also helpful to have a collection of nonlinear problems for which one is familiar with at least one explicit analytical solution. Initial integrals, which are also referred to as conservation laws, or, more generally speaking, Lyapunov functions, have the potential to serve as the foundation for subsequent testing and theoretical results. We hope that this will stimulate the reader's curiosity sufficiently to inspire them to investigate this topic further, despite the fact that we are only able to touch on these issues briefly due to space limits.

Differential equations

Within the context of a differential equation, the unknown variable is not a number but rather a function. It is necessary for the unknown functions or their derivatives to be linear in order for a linear differential equation to have any terms that are associated with each of these variables. In the context of this discussion, the term "linear" refers to the act of multiplying an unknown function by either a known function or a numerical value. None of the differential equations can be described as linear. One way to determine whether or not an equation is nonlinear is to search for nonlinear components in the solution to an equation that involves unknown functions or the product of their derivatives. This may be done in order to determine whether or not the equation is nonlinear. This is the simplest technique to determine whether the equation is. To provide an account of, in the event that

$$u'(t) = -a(t)u(t) + b(t),$$

Consider the following list of linear terms that are associated with the function that is unknown: A known function multiplies the unknown function by au, and the derivative of an unknown function is indicated by the symbol u 0. The derivative of an unknown function is raised to the power of one. Let me be honest:

$$u'(t) = u(t)(1 - u(t)),$$

Through the use of the equation −u 2, the function transforms into a nonlinear function when the unknown function is multiplied by itself.

.add to that,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0,$$

The fact that the unknown function is either a product of itself or one of its derivatives, in the instance of ux, is the source of the nonlinearity that manifests itself in the equation

For functions u(t) that only involve one variable, we make use of the notation u 0 or du/dt. We employ the notation ut or ut for functions that involve more than one variable. An further example of an equation that does not follow a linear pattern is

$$u'' + \sin(u) = 0,$$

Therefore, when the function is expanded using a Taylor series, the products of u are included within sin(u), which means that:

$$\sin(u) = u - \frac{1}{3}u^3 + \dots$$

II. OBJECTIVE

1. To study the algebraic techniques
2. To study the Nonlinear differential equations

III. RESEARCH METHODOLOGY

The purpose of this work is to resolve systems of nonlinear algebraic equations that are a result of the discretization of ordinary differential equations and partial differential equations. This will be accomplished via the use of algebraic techniques such as Picard iteration and Newton's method. The research then moves on to characterize the several kinds of nonlinear algebraic systems that were found via the use of implicit temporal discretization. This is accomplished by using both general and specialized systems, such as A(u)u = b(u) and F(u)=0, where uuu describes a vector of unknowns. This is done by using generic systems like A(u)u = b(u) among other specialized systems. Mathematical modeling makes use of the Crank-Nicolson scheme and other temporal discretization methods in order to represent these nonlinear differential equations, which then result in the production of systems of nonlinear equations. Picard iteration is utilized for systems that have a certain structure in order to enable term decoupling and to approximate nonlinear expressions by fixing them at values that were computed in the past. After that, a linear system is obtained, which can be solved multiple times until convergence is reached. When dealing with nonlinear systems that are more generic in nature, Newton's technique is utilized. This technique entails determining the Jacobian matrix of the system and linearly approximating the initial nonlinear system by the utilization of a Taylor expansion. Solving the linearized system is made possible through the process of iteratively updating the solution vector.

Nonlinear Algebraic Equation Systems:-

When an ordinary differential equation (ODE) or partial differential equation (PDE) is solved by employing implicit temporal discretization, the resulting system of nonlinear algebraic equations is compactly stated as

$$F(u) = 0,$$

The vector function F is represented by the equation F = (F0,... FN), and the vector of unknowns u is represented by the equation u = (u0,..., uN). Under the assumption that N is equal to two and that F0(u) is represented by the left-hand side of (18), the system that was discussed at the conclusion of Section 1.12 is compatible with this notation. F1(u) is the left-hand side of.

It is possible for the system of equations to take on a structure that is distinct when the nature of the underlying problem is taken into consideration; for example,

$$A(u)u = b(u),$$

Given a set of values u and b, where b is a matrix with dimensions (N + 1)×(N + 1), we can define A(u) as a matrix function of u.

In the next section, we will discuss systems such as F(u) = 0 and A(u)u = b(u), as well as how to use Picard iteration and Newton's approach. This talk is focussed around concepts and calculations that are relevant to the actual world. The book Kelley presents a number of theoretical topics, including results that are fairly broad in nature regarding the convergence properties of different techniques.

Picard iteration

The Picard iteration cannot be utilized for the solution of nonlinear equations unless a specific framework is utilized. In order to simplify the product, it is necessary to reduce A(u)u to A(u −)u and b(u) to b(u −). It is a(u)u. This simplifies the scenario where A(u)u equals b(u), which occurs rather frequently. To put it another way, we obtain a linear system for u by employing the most recent approximation that was obtained in A and b:

$$A(u^-)u = b(u^-)\,.$$

A relaxed iteration takes the form

$$A(u^-)u^* = b(u^-), \quad u = \omega u^* + (1-\omega)u^-$$

The implication of this is that we are able to solve an algebraic system of nonlinear equations by first dividing it into a number of linear systems. It is determined that the iteration is terminated if the change in the unknown or residual, which is represented as ||A(u)u − b(u)||, is deemed to be of sufficient magnitude.

Newton's method

When it comes to putting Newton's approach into reality, the most obvious place to begin is with the standard nonlinear vector equation F(u) = 0. The principle is the same as when dealing with a scalar equation, in which the linear function F† is derived by applying the Taylor expansion to the first two components of F. Following that, this function is applied in order to estimate the value of F with regard to a value u that has been provided. The combination of these two phrases is what happens when more than one variable is taken into account.

$$F(u^-) + J(u^-)\cdot(u-u^-),$$

In this case, J is defined as the Jacobian of F by

$$J_{i,j} = \frac{\partial F_i}{\partial u_j}\,.$$

Consequently, the initial nonlinear system is roughly represented by

$$\hat{F}(u) = F(u^-) + J(u^-)\cdot(u-u^-) = 0,$$

In order to solve this linear problem, there are two actions that can be taken: To begin, you need make the necessary adjustments to the equation u = u − + δu. This may be accomplished by locating the vector δu in reference to the equation Jδu = −F(u −). Adding a relaxation parameter is a straightforward process:

$$u = \omega(u^- + \delta u) + (1-\omega)u^- = u^- + \omega\delta u\,.$$

If a system has structure A(u), then the equation u = b(u) stays true.

$$F_i = \sum A_{i,k}(u)u_k - b_i(u),$$

It is possible to get it.

$$J_{i,j} = A_{i,j} + \sum_k \frac{\partial A_{i,k}}{\partial u_j}u_k - \frac{\partial b_i}{\partial u_j}\,.$$

Just like in the Picard iteration, we are aware that the Jacobian that is necessary for Newton's technique is denoted by the symbol A(u −). However, the differentiation process results in the addition of two more terms. For a linear system with three indices (∂Ai, k/∂uj) and two indices (∂bi/∂uj), it is possible to define it as A0 (u) for ∂A/∂u and b 0 (u) for ∂b/∂u, respectively.

$$(A + A'u - b')\delta u = -Au + b,$$

Or

$$(A(u^-) + A'(u^-)u^- - b'(u^-))\delta u = -A(u^-)u^- + b(u^-)\,.$$

The difference between the system that has been solved and each Picard iteration is made abundantly evident when the terms are rearranged:

$$\underbrace{A(u^-)(u^- + \delta u) - b(u^-)}_{\text{Picard system}} + \gamma(A'(u^-)u^- - b'(u^-))\delta u = 0\,.$$

Within this context, we have introduced a parameter denoted as γ. The Picard system is represented by γ = 0, whereas the Newton system is represented by γ = 1. A setting like this would be useful for software that gives users the ability to switch between multiple ways in a short amount of time.

Stopping criteria

The typical norm for vectors in geometry is . There are four often used termination criteria:

- ||u → u →|| ≤ u is the absolute value of the solution's change.
- In each iteration, u0 represents the initial value of u −, and the relative change in the solution is given by the inequality ||u − u −|| ≤ u||u0||.
- The absolute residual is defined as the absolute value of F(u) ≤ r.
- The relative residual is defined as the absolute value of F(u) ≤ r / ||F(u0)||.

The current iteration count, k, is stopped when it reaches a maximum value, k max, in order to prevent divergent iterations from continuing on indefinitely. This is done in order to prevent arbitrariness. On account of the fact that the relative criteria are not affected by the characteristic size of u, they are the ones that are applied the most frequently. On the other hand, if the initial value of the iteration start is really near to the solution, then use the relative criteria to compel an excessive decrease in the error measure could provide results that are deceptive. This is because the relative criterion is dependent on the original start

value. These are the circumstances in which the standards that are absolute prove to be more effective. It is a frequent practice at this point in time to combine measurements of absolute and relative residual size, as can be seen in the example given in

$$||F(u)|| \leq \epsilon_{rr}||F(u_0)|| + \epsilon_{ra},$$

In this particular instance, the relative criterion's tolerance is denoted by the symbol rr, whereas the absolute criterion's tolerance is denoted by the symbol ra. Given that the term ||F(u0)|| is quite tiny at the previous time level and that ra is the prevalent tolerance, it may be deduced that the initial prediction for the iteration was quite accurate. For all other situations, the relative criterion and rr||F(u0)|| are the most important factors to consider.
We are able to produce a combined absolute and relative measure of the change in the solution by using the change in solution as our criteria. This measure is as follows:

$$||\delta u|| \leq \epsilon_{ur}||u_0|| + \epsilon_{ua},$$

At the end of the day, the criteria for termination may be stated as after all of the factors, such as the residual, the solution change, and the maximum number of repetitions, has been taken into consideration

$$||F(u)|| \leq \epsilon_{rr}||F(u_0)|| + \epsilon_{ra} \quad \text{or} \quad ||\delta u|| \leq \epsilon_{ur}||u_0|| + \epsilon_{ua} \quad \text{or} \quad k > k_{\max}.$$

Nonlinear ordinary differential equation (ODE) model applied to epidemiology
A 2 × 2 ODE system is the most basic representation of how a disease, like the flu, might spread.

$$S' = -\beta SI,$$
$$I' = \beta SI - \nu I,$$

where S(t) represents the potential number of infected individuals and I(t) represents the number of susceptible individuals. In addition to the initial conditions S(0) and I(0), the constants β > 0 and v > 0 are required to be provided.
In order to achieve the implicit temporal discretization in the variables S n+ 1 and me n+ 1, a Crank-Nicolson technique is used. This approach ultimately results in a system of nonlinear algebraic equations comprised of two by two equations.

$$\frac{S^{n+1} - S^n}{\Delta t} = -\beta[SI]^{n+\frac{1}{2}} \approx -\frac{\beta}{2}(S^n I^n + S^{n+1} I^{n+1}),$$
$$\frac{I^{n+1} - I^n}{\Delta t} = \beta[SI]^{n+\frac{1}{2}} - \nu I^{n+\frac{1}{2}} \approx \frac{\beta}{2}(S^n I^n + S^{n+1} I^{n+1}) - \frac{\nu}{2}(I^n + I^{n+1})$$

If we replace S with Sn+1, Sn with Sn, I with In+1, and In with I(1), we may rewrite the system as

$$F_S(S, I) = S - S^{(1)} + \frac{1}{2}\Delta t\beta(S^{(1)} I^{(1)} + SI) = 0,$$
$$F_I(S, I) = I - I^{(1)} - \frac{1}{2}\Delta t\beta(S^{(1)} I^{(1)} + SI) + \frac{1}{2}\Delta t\nu(I^{(1)} + I) = 0.$$

An example of a Picard iteration would be the assumption that approximations S − and I − to S and I are available in this particular circumstance. Once the two equations are partitioned into I →S in the FS = 0 equation and S →I in the FI = 0 equation, the nonlinear term SI may be transformed into a linear expression. This is one method for achieving this transformation. The usage in question is one of the many possibilities. By solving the equations that are produced as a consequence of the process, it is possible to deduce the linear equations that make up the output of the process. It is possible to get these equations by making use of the variables S and I.

$$S = \frac{S^{(1)} - \frac{1}{2}\Delta t\beta S^{(1)} I^{(1)}}{1 + \frac{1}{2}\Delta t\beta I^-},$$
$$I = \frac{I^{(1)} + \frac{1}{2}\Delta t\beta S^{(1)} I^{(1)} - \frac{1}{2}\Delta t\nu I^{(1)}}{1 - \frac{1}{2}\Delta t\beta S^- + \frac{1}{2}\Delta t\nu}.$$

In order to go on with a new iteration, it is essential to perform an update on both S − ← S and I − ↞ I. We may use Newton's method to express the nonlinear system (28)–(29) as F(u) = 0 by substituting u = (S, I) for FS and FI, respectively. This will allow us to accurately represent the system. The equation may be used to demonstrate that the system is nonlinear. The process by which a Jacobian is transformed into

$$J = \begin{pmatrix} \frac{\partial}{\partial S}F_S & \frac{\partial}{\partial I}F_S \\ \frac{\partial}{\partial S}F_I & \frac{\partial}{\partial I}F_I \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{2}\Delta t\beta I & \frac{1}{2}\Delta t\beta S \\ -\frac{1}{2}\Delta t\beta I & 1 - \frac{1}{2}\Delta t\beta S + \frac{1}{2}\Delta t\nu \end{pmatrix}$$

Additionally, it is necessary to solve the Newton system J(u −)δu = −F(u −) at the conclusion of each cycle.

$$\begin{pmatrix} 1 + \frac{1}{2}\Delta t\beta I^- & \frac{1}{2}\Delta t\beta S^- \\ -\frac{1}{2}\Delta t\beta I^- & 1 - \frac{1}{2}\Delta t\beta S^- + \frac{1}{2}\Delta t\nu \end{pmatrix} \begin{pmatrix} \delta S \\ \delta I \end{pmatrix} = - \begin{pmatrix} S^- - S^{(1)} + \frac{1}{2}\Delta t\beta(S^{(1)} I^{(1)} + S^- I^-) \\ I^- - I^{(1)} - \frac{1}{2}\Delta t\beta(S^{(1)} I^{(1)} + S^- I^-) + \frac{1}{2}\Delta t\nu(I^{(1)} + I^-) \end{pmatrix}$$

Linearization of Differential Equations

The focus has shifted from ordinary differential equations (ODEs) to nonlinear partial differential equations (PDEs), as well as the application of the methods that were covered earlier. The model problem is represented by an equation for nonlinear diffusion written as u(x, t):

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla \cdot (\alpha(u)\nabla u) + f(u), && \boldsymbol{x} \in \Omega,\ t \in (0,T], \\ -\alpha(u)\frac{\partial u}{\partial n} &= g, && \boldsymbol{x} \in \partial\Omega_N,\ t \in (0,T], \\ u &= u_0, && \boldsymbol{x} \in \partial\Omega_D,\ t \in (0,T]\,. \end{aligned}$$

The time-discrete discretization of the issue will be followed by a demonstration of how to linearize the time-discrete partial differential equation (PDE) problem "at the PDE level." First, we will make this matter more discrete with regard to the passage of time. Therefore, the nonlinear stationary PDE issue at each time step may be reduced to a sequence of linear PDE problems, each of which can be addressed using any linear PDE technique. This implies that the problem can be handled in a linear fashion. Through the use of this method, it is feasible to avoid the need of solving nonlinear algebraic equation systems. There is also the possibility that we might stick to the conventional way of thinking and discretize the nonlinear issue in space and time. This is an alternate viewpoint that might be considered. Once everything was accomplished, we would be able to put the strategies into practice in order to find answers to the nonlinear algebraic equations that were generated at each time level. As a result of the fact that the two methods are mathematically equivalent to one another, there is often no choice between them in terms of the computer efficiency that they provide. Let us have faith that the examples that are offered below will assist in elucidating the issues that were covered previously.

It is possible that the nonlinearities in the PDE may be easily managed by using a method such as the Forward Euler approach, which requires explicit time integration for the value of thirty. This will be successful so long as we continue to take that attitude.

$$[D_t^+ u = \nabla \cdot (\alpha(u)\nabla u) + f(u)]^n,$$

or written out,

$$\frac{u^{n+1} - u^n}{\Delta t} = \nabla \cdot (\alpha(u^n)\nabla u^n) + f(u^n),$$

A linear equation that includes the unknown u n+1 is considered to be the solution to this problem.

$$u^{n+1} = u^n + \Delta t \nabla \cdot (\alpha(u^n)\nabla u^n) + \Delta t f(u^n)\,.$$

One of the limitations of this discretization that has downsides is the standard 2nd-order finite difference discretization in three-dimensional space. This discretization has mesh cell sizes of h = Δx = Δy = Δz. Additionally, in the scenario when the value of f is equal to zero, the stringent stability criterion Δt < h 2/(6 max α) is applicable. The following is an example of a backward-Euler setup that you should take into consideration: 30 minutes of reading time

$$[D_t^- u = \nabla \cdot (\alpha(u)\nabla u) + f(u)]^n\,.$$

Written out,

$$\frac{u^n - u^{n-1}}{\Delta t} = \nabla \cdot (\alpha(u^n)\nabla u^n) + f(u^n)\,.$$

This particular sort of differential equation, known as a nonlinear partial differential equation (PDE), is used to simulate the function u n(x), which is something that is unknown. It is possible to consider this specific kind of partial differential equation (PDE) as a time-independent PDE, which is distinguished by a well-known function u n−1 (x). There is a counter for iterations that is introduced at the Picard iteration. This number is symbolized by the letter k. The linearization of the ∬• (α(u n)≒u n) component in iteration k+1 is a technique that is often used. Through the use of the u n,k approximation, which was discovered previously in the explanation of the diffusion coefficient: α(u n,k), this objective is accomplished. When it comes to the f(u n,k) notation, which is used to denote the nonlinear source term, the same line of reasoning applies. Therefore, it is possible that the requirements of the linear partial differential equation may be satisfied by the missing function u n,k+1.

$$\frac{u^{n,k+1} - u^{n-1}}{\Delta t} = \nabla \cdot (\alpha(u^{n,k})\nabla u^{n,k+1}) + f(u^{n,k})\,.$$

When considering this particular time level, it is possible to make an approximate estimation of the Picard iteration by using the outcome from the time level before this one, which is u n,0 = u n−1. There is also the possibility of using the notation that is simpler to put into practice. According to this notation, the variable u represents the unknown that we are attempting to discover (u n,k+1 above), whereas the number u − represents the most recent computation (u n,k above). Furthermore, it is worth noting that the function u (1) is referred to as the unknown function at the time level that came before it, which is known as u n−1.

At this time, the Picard iteration-solvable PDE seems to be becoming more and more

$$\frac{u - u^{(1)}}{\Delta t} = \nabla \cdot (\alpha(u^-)\nabla u) + f(u^-)\,.$$

After each iteration, we make sure that u − is immediately updated to reflect the current value of u. The starting value for iteration u is u − = u (1). Continuing from the time level before the current one, this operation is carried out all the way up to the current time level.

By using Newton's technique and the Euler system in the other direction. At time level n, our objective is to locate a stationary solution to the partial differential equation that describes the relationship between two locations. What we want to do is this. This was done by the use of Picard iterations, which were covered in the part that came before this one. The use of Newton's method in a novel manner is yet another strategy that might be considered. Despite the fact that Newton's method is often provided for solving systems of algebraic equations, it is feasible to further develop it such that it may be used to solve partial differential equations (PDEs). Taylor expansions are a method that may be used to achieve linearity. This is the most recent

approximation of the indeterminate u n, which is represented by the notation u n,k. We are gathering information in an effort to produce a form estimate that is more accurate.

$$u^n = u^{n,k} + \delta u\,.$$

Given equation, it is possible to estimate the value of u n by extending the nonlinearities using Taylor and maintaining just the linear components in the variable δu. Following that, we can come across a new approximation by using equation (36), which requires us to solve a linear partial differential equation (PDE) for the adjustment δu.

$$u^{n,k+1} = u^{n,k} + \delta u$$

as the letter u'n should be placed. The objective u n will be attained by the sequence u n,k+1, where k is equal to 0, 1,..., provided that everything continues to proceed as planned. There is no doubt that this will take happen. By discretizing it using the Backward Euler technique, it is possible to resolve all of the mathematical subtleties that are associated with the nonlinear diffusion partial differential equation (PDE).

$$\frac{u^{n,k} + \delta u - u^{n-1}}{\Delta t} = \nabla \cdot (\alpha(u^{n,k} + \delta u)\nabla(u^{n,k} + \delta u)) + f(u^{n,k} + \delta u)$$

The application of Taylor expansion may be made to both α(u n,k + δu) and f(u n,k + δu) according to the following:

$$\alpha(u^{n,k} + \delta u) = \alpha(u^{n,k}) + \frac{d\alpha}{du}(u^{n,k})\delta u + \mathcal{O}(\delta u^2) \approx \alpha(u^{n,k}) + \alpha'(u^{n,k})\delta u,$$

$$f(u^{n,k} + \delta u) = f(u^{n,k}) + \frac{df}{du}(u^{n,k})\delta u + \mathcal{O}(\delta u^2) \approx f(u^{n,k}) + f'(u^{n,k})\delta u\,.$$

A solution is obtained by incorporating the linear approximations of α and f into equation:

$$\begin{aligned}\frac{u^{n,k} + \delta u - u^{n-1}}{\Delta t} = {} & \nabla \cdot (\alpha(u^{n,k})\nabla u^{n,k}) + f(u^{n,k}) + \\ & \nabla \cdot (\alpha(u^{n,k})\nabla \delta u) + \nabla \cdot (\alpha'(u^{n,k})\delta u \nabla u^{n,k}) + \\ & \nabla \cdot (\alpha'(u^{n,k})\delta u \nabla \delta u) + f'(u^{n,k})\delta u\,.\end{aligned}$$

The inclusion of the component α 0 (u n,k)δu∇δu, which is of order δu2, is not included in our analysis. This is because we assume that the adjustment δu is very minor (δu δu2). Through the process of rearranging the equation, it is possible to derive the simplified form of the partial differential equation (PDE) for the variable δu.

$$\delta F(\delta u; u^{n,k}) = -F(u^{n,k}),$$

Where

$$F(u^{n,k}) = \frac{u^{n,k} - u^{n-1}}{\Delta t} - \nabla \cdot (\alpha(u^{n,k})\nabla u^{n,k}) - f(u^{n,k}),$$

$$\begin{aligned}\delta F(\delta u; u^{n,k}) = {} & \frac{1}{\Delta t}\delta u - \nabla \cdot (\alpha(u^{n,k})\nabla \delta u) - \\ & \nabla \cdot (\alpha'(u^{n,k})\delta u \nabla u^{n,k}) - f'(u^{n,k})\delta u\,.\end{aligned}$$

Taking into consideration that δF is a linear function of δu and that F only comprises components that are known, it is of utmost importance to admit that the partial differential equation (PDE) for δu is linear.

By rewriting the equation u n,k + δu as u n,k+1, it is also feasible to rewrite the probability density function (PDE) for δu in a slightly different way. This is accomplished via the process of rewriting the expression. As a consequence of this, we are in a position to assess it in connection to the Picard iteration.

$$\begin{aligned}\frac{u^{n,k+1} - u^{n-1}}{\Delta t} = {} & \nabla \cdot (\alpha(u^{n,k})\nabla u^{n,k+1}) + f(u^{n,k}) \\ & + \nabla \cdot (\alpha'(u^{n,k})\delta u \nabla u^{n,k}) + f'(u^{n,k})\delta u\,.\end{aligned}$$

Pay attention to the fact that the first line reflects the same partial differential equation (PDE) that was used in the Picard iteration before. It is essential that you keep this knowledge in mind. Differentiations are the foundation of Newton's approach, which results in the phrases that are presented in the following paragraphs.

We want to use the notation u for the value of u n, u − for the value of u n,k, and u (1) for the value of u n−1 throughout the process of coding. Because of this, the equations for F and δF may be stated in a more straightforward manner as

$$F(u^-) = \frac{u^- - u^{(1)}}{\Delta t} - \nabla \cdot (\alpha(u^-)\nabla u^-) - f(u^-),$$

$$\begin{aligned}\delta F(\delta u; u^-) = {} & \frac{1}{\Delta t}\delta u - \nabla \cdot (\alpha(u^-)\nabla \delta u) - \\ & \nabla \cdot (\alpha'(u^-)\delta u \nabla u^-) - f'(u^-)\delta u\,.\end{aligned}$$

As a result of the addition of the derivative terms derived from the Newton approach and the Picard iteration terms, the form that orders the PDE is transformed into

$$\begin{aligned}\frac{u - u^{(1)}}{\Delta t} = {} & \nabla \cdot (\alpha(u^-)\nabla u) + f(u^-) + \\ & \gamma(\nabla \cdot (\alpha'(u^-)(u - u^-)\nabla u^-) + f'(u^-)(u - u^-))\,.\end{aligned}$$

When we check the Picard and complete Newton versions, we find that the variable γ is represented by a value of 0 and a value of 1 respectively.

In order to create the linearized partial differential equation (PDE) for δu, it may be more appealing to use the notation that is more straightforward. This is because the notation is more straightforward. In order to achieve, our initial step is the incorporation of the equation u n = u − + δu.

$$\frac{u^- + \delta u - u^{n-1}}{\Delta t} = \nabla \cdot (\alpha(u^- + \delta u)\nabla(u^- + \delta u)) + f(u^- + \delta u)$$

Taylor expanding,

$$\alpha(u^- + \delta u) \approx \alpha(u^-) + \alpha'(u^-)\delta u,$$
$$f(u^- + \delta u) \approx f(u^-) + f'(u^-)\delta u,$$

The incorporation of these equations results in a less cluttered partial differential equation (PDE) for the variable δu.

$$\frac{u^- + \delta u - u^{n-1}}{\Delta t} = \nabla \cdot (\alpha(u^-)\nabla u^-) + f(u^-) + \nabla \cdot (\alpha(u^-)\nabla \delta u) + \nabla \cdot (\alpha'(u^-)\delta u \nabla u^-) + \nabla \cdot (\alpha'(u^-)\delta u \nabla \delta u) + f'(u^-)\delta u\,.$$

Crank-Nicolson discretization

When discretizing using the Crank-Nicolson method, a centered difference is applied at tn+1/2:

$$[D_t u = \nabla \cdot (\alpha(u)\nabla u) + f(u)]^{n+\frac{1}{2}}$$

Due to the fact that u is unknown at tn+1/2, we are required to define the right-hand terms by utilizing u n and u n+1, both of which are uncertain. The most reliable approach is to use arithmetic averages.

$$u^{n+\frac{1}{2}} \approx \frac{1}{2}(u^n + u^{n+1})$$

However, when dealing with nonlinear factors, there are a number of different techniques to determine the mathematical mean, including the following:

$$[f(u)]^{n+\frac{1}{2}} \approx f(\frac{1}{2}(u^n + u^{n+1})) = [f(\overline{u}^t)]^{n+\frac{1}{2}},$$
$$[f(u)]^{n+\frac{1}{2}} \approx \frac{1}{2}(f(u^n) + f(u^{n+1})) = [\overline{f(u)}^t]^{n+\frac{1}{2}},$$
$$[\alpha(u)\nabla u]^{n+\frac{1}{2}} \approx \alpha(\frac{1}{2}(u^n + u^{n+1}))\nabla(\frac{1}{2}(u^n + u^{n+1})) = [\alpha(\overline{u}^t)\nabla \overline{u}^t]^{n+\frac{1}{2}},$$
$$[\alpha(u)\nabla u]^{n+\frac{1}{2}} \approx \frac{1}{2}(\alpha(u^n) + \alpha(u^{n+1}))\nabla(\frac{1}{2}(u^n + u^{n+1})) = [\overline{\alpha(u)}^t \nabla \overline{u}^t]^{n+\frac{1}{2}},$$
$$[\alpha(u)\nabla u]^{n+\frac{1}{2}} \approx \frac{1}{2}(\alpha(u^n)\nabla u^n + \alpha(u^{n+1})\nabla u^{n+1}) = [\overline{\alpha(u)\nabla u}^t]^{n+\frac{1}{2}}\,.$$

It is not quite clear if computing the arithmetic mean of products or the products of arithmetic means produces considerably different results in terms of accuracy. In relation to this topic, there is a tremendous degree of uncertainty and misunderstanding. As stated in Exercise 6, which is specifically concerned with this particular topic, the two examples demonstrate that the approximation requires O(Δt 2) to be performed.

IV. CONCLUSION

In a wide variety of scientific and engineering fields, nonlinear differential equations are frequently encountered, and algebraic methods have become indispensable for the resolution of these equations. Several fields of study, such as quantum physics, fluid mechanics, wave propagation, and population dynamics, rely on these equations to provide an explanation for complex occurrences. For the purpose of simplifying equations for nonlinear systems and, in many cases, providing analytical or semi-analytical solutions, algebraic approaches are invaluable. This is due to the fact that classical methods are unable to identify accurate solutions for such systems. Some of the most prominent methods include the Lie group method, the Hirota bilinear method, and the Painlevé analysis. Each of these methods offers a unique approach to solving issues that are similar to one another. The primary objective of the Lie group technique, which enables reductions that simplify complex systems, is to accomplish the discovery of symmetry in differential equations. On the other hand, the Hirota bilinear technique is excellent in locating soliton solutions, which are essential for understanding the dynamics of nonlinear waves. The Painlevé analysis provides insight into the process of locating integrable equations, which is important when it comes to the classification of differential equations according to their solvability. When it comes to evaluating the stability and qualitative behavior of solutions in nonlinear systems, these techniques are not only excellent for locating precise solutions, but they are also excellent at assessing the stability of solutions. It is possible to develop numerical or approximative solutions with their assistance in situations where exact solutions are not feasible. In addition, the structure and behavior of nonlinear equations can be better understood with the use of algebraic approaches (also known as algebraic strategies). Algebraic methods continue to be critically important in the advancement of both theoretical and applied research across a wide range of scientific fields. This is because of the practical solutions they provide and the insights they provide into the behavior of nonlinear differential equations.

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