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Exact Modules under Change of Rings

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Abstract: Throughout this paper I have proved some results which are establishing the preservance of exactness property under change of ring. First of all If R and S are two rings such that $_R M$ and $_S M$ are two left modules then for a surjective homomorphism Ø: S → R, if $_R M$ is exact then $_S M$ is exact.Then I have established the converse of above result as : Let f:R→S be a surjective ring homomorphism and let M be an abelian group that is simultaneously a left R-

module and a left S-module and if $_R M$ is an exact module then $_S M$ is exact. Later I have defined weakly exact module and with the result if $_R M$ be any exact left R-module, $_S M$ be any other module and f:R→ S be a homomorphism suppose $M_1{}'$ be a maximal submodule of $_S M_{End(_S M)}$ then $_R M_1{}'_{End(_R M)}$ is weakly exact bimodule. At last I have the result that if $_R M$ be any module of finite length and if for any n>0, $M^{(n)}$ is exact then $_R M$ is exact.

Keywords: change of rings, homomorphism, surjective homomorphism, Balanced bimodules.

INTRODUCTION

Here I have taken some definitions from [2] and [1]

Change of Rings:

Def1: If R and S be any two rings such that $_S M$ be any left S-modules and f:R→S

is a ring homomorphism . If the structure $_S M$ obtained from the ring

homomorphism

$\lambda: S \to End^l(M)$

then composition
$\lambda o f: R \to End^l(M)$

induces left R-structure on M. Here the scalar multiplication is given by

$(r, x) \mapsto f(r)$. Thus $_S M$ is a left R-module $_R M$ with

$rx = f(r)x \qquad (r \in R, x \in M)$

Exact bimodule:

Def2: Let ${}_R M_S$ be any bimodule such that it has a composition series

${}_R M_S = M_0 \geq M_1 \geq M_2 \geq \; . \; . \; . \; . \; . \; . \; . \; . \; . \geq M_n = 0$

With balanced composition factors ${}_R (M_{i-1}/M_i)_S$ is called an exact bimodule and that a ring R is an exact ring in case the regular bimodule ${}_R R_R$ is exact . A one sided module ${}_R M$ is exact if the bimodule ${}_R M_{End({}_R M)}$ is an exact bimodule.

In this paper we studied the property of exactness under the change of rings, for this we will define the function between two rings and we will show that functor will preserve the property of exactness between two categories for different form of homomorphism such as epimorphism monomorphism and isomorphism. At last we will show that if ${}_R M^{(n)}$ is exact for any n>0 then ${}_R M$ is exact.

Here we have taken Rings and various forms of Homomorphisms without loss of generality in any manner. Also here we have a assumption that all rings have unity and all modules are unitary.

Theorem1: Let R and S be two rings such that ${}_R M$ and ${}_S M$ are two left modules, if $\varnothing: S \rightarrow R$ is surjective and ${}_R M$ is exact then ${}_S M$ is exact.

Proof: Given that ${}_R M$ be any exact left R-modules, then by the definition of exact left R-module for T=End(${}_R M$) the bimodule ${}_R M_T$ is exact i.e. it has a composition series

${}_R M_T = M_0 \geq M_1 \geq M_2 \geq . \; . \; . \quad . \; . \; . \quad . \; . \; . \geq M_n = (0)$

such that composition factors

${}_R M_{i-1}/M_{i\,T}$ i=1, 2, 3 . . . n

are balanced, so the ring homomorphism

$\lambda: R \rightarrow End(M_{i-1}/M_{i\,T})$ and $\rho: T \rightarrow End({}_R M_{i-1}/M_i)$

are surjective, if S is any other rings such that

$$\text{Ø}:S \to R$$

Is surjective ring homomorphism and since $_{S}M$ is given left S-modules then via surjections

$\lambda o\text{Ø}:S \to End(M_{i-1}/M_{i\ T})$ and $\rho:End({}_{\phi(S)}M_{i-1}/M_{i}) \to End({}_{\phi(S)}M_{i-1}/M_{i})$

$_{S}M_{i-1}/M_{i\ End(_{S}M)}$ i=1,2,3 . . . n are balanced then bimodule

$_{S}M_{End(_{S}M)}$ is exact and consequently $_{S}M$ is exact.

Corollary1.1: Let R and S be any two rings such that $R \cong S$ then left R-module M is exact iff left S-modules M is exact.

Proof: Let $_{R}M$ be any exact left R-modules and $_{S}M$ be a left S-module structure on M since given that $R \cong S$ then obviously $_{R}M \cong {}_{S}M$ therefore $_{R}M$ is exact iff $_{S}M$ is exact.

Corollary1.2: Let $_{R}M$ be any exact module and S be any other ring such that $_{S}M$ be left S-module if bimodule $_{S}R_{R}$ is balanced then $_{S}M$ is exact.

Proof: Given that $_{R}M$ be any exact module and S be any other ring such that $_{S}M$ be left S-modules since here bimodule $_{S}R_{R}$ is given balanced then ring homomorphisms

$\lambda: S \to End(R_{R})$ and $\rho:R \to End({}_{S}R)$

are surjective then from Proposition 4.11[2] P 60, the ring homomorphism

$\lambda: S \to R$ is surjective since here $_{R}M$ is given exact then from Th:1, $_{S}M$ is exact.

Theorem2: Let f:R→S be a surjective ring homomorphism and let M be an abelian group that is simultaneously a left R-module and a left S-module if $_{R}M$ is an exact module then $_{S}M$ is exact.

Proof: Let $_{R}M$ be any exact left R-modules then by definition bimodule

${}_R M_{End({}_R M)}$ has a composition series

${}_R M_{End({}_R M)} = M_0 \geq M_1 \geq M_2 \geq \ . \ . \ . \quad . \ . \ . \quad . \ . \ . \quad \geq M_n = (0)$

such that composition factors ${}_R M_{i-1}/M_i$ are balanced then

$\lambda: R \rightarrow End(M_{i-1}/M_{i\ End({}_R M_{i-1}/M_i)})$ and $\rho: End({}_R M_{i-1}/M_i) \rightarrow End({}_R M_{i-1}/M_i)$

are surjective ring homomorphisms suppose S is other ring such that f: R→ S is

surjection and if the structure ${}_S M$ is obtained from the ring homomorphism

Ø:S→ End^l (M), then

Øof:R → End^l (M)

induces a left R-structure on M. Here the scalar multiplication is given by

(r , x)↦ f(r)x. Thus ${}_S M$ is a left R-module ${}_R M$ with

rx=f(r)x ($r \in R$, $x \in M$)

since f:R→ S is a surjection then clearly from Propsition2.11 [2] P 35, the

composition series for ${}_R M_{End({}_R M)}$ is also be the composition series for ${}_S M_{End({}_S M)}$

then we have a commutative diagram

$$
\begin{array}{ccc}
R & \xrightarrow{\lambda} & End(M_{i-1}/M_{i\ End({}_R M_{i-1}/M_i)}) \\
f\downarrow & & \downarrow \\
S & \xrightarrow{\psi} & End(M_{i-1}/M_{i\ End({}_S M_{i-1}/M_i)})
\end{array}
$$

since f:R→ S is surjection so from 4.8[2] P 59,

End(${}_S M$)=End(${}_R M$)

and similarly

$End({}_S M_{i-1}/M_i) = End({}_R M_{i-1}/M_i)$

thus g is an isomorphism then goλ=ψof

is also be a surjection since f is given surjection then ψ is also be a surjection and

similarly

$\upsilon: End({}_S M_{i-1}/M_i) \rightarrow End({}_S M_{i-1}/M_i)$

is also be a surjection therefore left S-module ${}_S M$ is also exact.

Weakly Exact module:

Def3: Let ${}_R M$ be any module, then ${}_R M$ is called weakly exact module in case

bimodule ${}_R M_{End({}_R M)}$ has a composition series

$${}_R M_{End({}_R M)} = M_0 \geq M_1 \geq M_2 \geq \ \ldots \ \ldots \ \ldots \ \geq M_n = (0)$$

such that composition factors ${}_R (M_{i-1}/M_i)$ where i=2,3 . . . . n are balanced.

Corollary 3: Let ${}_R M$ be any exact left R-module ${}_S M$ be any other module and

f:R→ S be a homomorphism suppose $M_1{}'$ be a maximal submodule of ${}_S M_{End({}_S M)}$

then ${}_R M_1{}'_{End({}_R M)}$ is weakly exact bimodule.

Proof: Given that ${}_R M$ be any exact left R-module then by definition bimodule

${}_R M_{End({}_R M)}$ has a composition series

$${}_R M_{End({}_R M)} = M_0 \geq M_1 \geq M_2 \geq \ \ldots \ \ldots \ \ldots \ \geq M_n = (0)$$

with balanced composition factors ${}_R (M_{i-1}/M_i)$

now suppose f:R→ S is a homomorphism and ${}_S M$ be any other module then

there are four modules

${}_S M$, ${}_R M$, ${}_{f(R)} M$, ${}_Z M$

and each submodule of any one of these is a submodule of each subsequent one

let $M_1{}'$ be a maximal submodule of ${}_S M_{End({}_S M)}$ then $M_1{}' \leq {}_R M$ and from

Proposition 11.1[2] P 134, ${}_R M_1{}'_{End({}_R M)}$ has a composition series

$${}_R M_1{}'_{End({}_R M)} = M_1{}' \geq M_2{}' \geq \ \ldots \ \ldots \ \ldots \ \geq M_{n'}{}' = (0)$$

and we have

$${}_R M_{End({}_R M)} \geq M_1{}' \geq M_2{}' \geq \ \ldots \ \ldots \ \ldots \ \geq M_{n'}{}' = (0)$$

then

$${}_R M_{End({}_R M)} \cap M_1{}' = M_0 \cap M_1{}' \geq M_1 \cap M_1{}' \geq M_2 \cap M_1{}' \geq \ \ldots \geq M_n \cap M_1{}' = (0)$$

Suppose i is the number such that $M_{i-1} \leq M_1{}'$, then

$${}_R M_1{}'{}_{End({}_R M)} = M_1{}' \geq M_1{}' \geq M_1{}' \geq \ \ldots \ \geq M_{i-1} \geq M_i \geq \ \ldots \ \geq M_n = (0)$$

then we can write above composition series as

$${}_R M_1{}'{}_{End({}_R M)} = M_1{}' \geq M_{i-1} \geq M_i \geq \ \ldots \ \geq M_n = (0).$$

Since here ${}_R M$ is given exact so ${}_R (M_{i-1}/M_i)$ is balance and consequently

${}_R M_1{}'{}_{End({}_R M)}$ is a weakly exact bimodule.

Corollary4: Let ${}_R M_S$ be balanced bimodule then bimodule ${}_R M_S$ is exact iff both ${}_R M$ and M_S are exact.

Proof: Given that ${}_R M_S$ be balanced left R and right S- bimodule then there are two surjections

$\lambda: R \to End(M_S)$ and $\rho: S \to End({}_R M)$

since here ${}_R M_S$ is given exact then by definition there is a composition series

$${}_R M_S = M_0 \geq M_1 \geq M_2 \geq \ \ldots \ \ldots \ \ldots \ \geq M_n = (0)$$

with balanced composition factors ${}_R M_{i-1}/M_{i\ S}$ then by the surjection we have

$\lambda(R) = End(M_S)$, $\rho(S) = End({}_R M)$

if given that ${}_R M$ and M_S be two left and right module respectively and by definition, ${}_R M$ and M_S are exact in case bimodule ${}_R M_{End({}_R M)}$ and ${}_{End(M_S)} M_S$ are exact now since both λ and ρ are surjection then we have

$${}_R M_S = {}_{\lambda(R)} M_S = {}_{End(M_S)} M_S \quad \text{and} \quad {}_R M_S = {}_R M_{\rho(S)} = {}_R M_{End({}_R M)}$$

and here ${}_R M_S$ is given exact therefore ${}_R M_{End({}_R M)}$ and ${}_{End(M_S)} M_S$ are exact and consequently ${}_R M$ and M_S are exact.

Conversely let ${}_R M$ and M_S are two exact modules then from definition, bimodule ${}_R M_{End({}_R M)}$ and ${}_{End(M_S)} M_S$ are exact now since ${}_R M_S$ is given balanced then we have

$_R M_{End(_R M)} = {}_R M_{\rho(S)} = {}_R M_S$ and $_{End(M_S)} M_S = {}_{\lambda(R)} M_S = {}_R M_S$

here since $_R M_{End(_R M)}$ and $_{End(M_S)} M_S$ are given exact so $_R M_S$ is exact.

Theorem5: Let $_R M$ be any module of finite length and if for any n>0, $M^{(n)}$ is exact then $_R M$ is exact.

Proof: Let M be any left R-module such that $_R M$ is of finite length then $_R M$ has a composition series so bimodule $_R M_{End(_R M)}$ has composition series

$_R M_{End(_R M)} = M_0 \geq M_1 \geq M_2 \geq \ . \ . \ . \quad . \ . \ . \quad . \ . \ . \quad \geq M_n = (0)$

which has composition factors $_R M_{i-1}/M_i$, now for any n>0, $_R M_{End(_R M^{(n)})}$ has a composition series

$_R M_{End(_R M^{(n)})} = M_0{}^{(n)} \geq M_1{}^{(n)} \geq M_2{}^{(n)} \geq \ . \ . \ . \quad . \ . \ . \quad . \ . \ . \quad \geq M_n{}^{(n)} = (0)$

with composition factors

$_R (M_{i-1}{}^{(n)}/M_i{}^{(n)}) \cong {}_R (M_{i-1}/M_i)^{(n)}$

since here $_R M^{(n)}$ is given exact so $_R (M_{i-1}{}^{(n)}/M_i{}^{(n)})$ and consequently $_R (M_{i-1}/M_i)^{(n)}$ is balanced now from exercise 4.1(13)[3] P 366, $_R (M_{i-1}/M_i)$ is also balanced and therefore $_R$ M is also be an exact module.

REFERENCES

[1] WEIMIN XUE: Exact Modules and Serial Rings, Journal of Algebra 134, 209-221 [1990]

[2] F. W. ANDERSON AND K. R. FULLER: Rings and Catagories of modules, New York Springer-Verlag Inc. 1973

[3] LOUIS HALLE ROWEN: Ring Theory

[4] T. Y. LAM: A first course in non commutative rings Springer-Verlag