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On Certain Hamiltonian Properties of WK-Recursive Networks

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Abstract

Among discrete mathematics, graph theory is a highly popular and interesting field. In 1736, *E.Euler* published the first known article on graph theory, which described the seven bridges of Königsberg.If every pair of a graph's vertices has a Hamiltonian path connecting them, the graph is said to be Hamilton-connected. It is an NP-complete task to determine if a graph is Hamilton-connected. We demonstrate the Hamiltonian connection between the line graphs of the generalized Petersen, antiprism, and wheel graphs by using these proof strategies. Incorporating it with some existing

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1 Introduction

Let $G = (V, E)$ be a non-empty graph of order $n \geq 2$, and l is a positive integer, We denote by V^l is the set of words of length l on alphabet V . The letters of a word u of length l are denoted by u_1, u_2, u_3 u_l. The concatenation of two words u and v is denoted by uv. Klavzar and Milutinovic introduced in [1] the graph $S(K_n, l)$, $l \geq 1$, whose vertex set V^l, where $\{u, v\}$ is an edge if and only if there exists $i \in \{1, 2, l\}$ such that

(i) $u_j = v_j$, if $j < i$

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(ii) $u_i \neq v_i$

(iii) $u_j = v_i$ and $v_j = u_i$ if $j > i$

As noted [2], in a compact form, the edge sets can be described as

$\{\{wu_i u_j^{d-1}, wu_j u_i^{d-1}\} : u_i, u_j \in V, i \neq j; d \in [t]; w \in V^{l-d}\}$ The graph $S(K_3, l)$ is isomorphic to the graph of the Tower of Hanoi with l disks [1]. Later, those graphs have been called Sierpinski graphs in [3] and they were studied by now from numerous points of view. For instance, the authors of [4] studied identifying codes, locating- dominating codes, and total-dominating codes in Sierpin- ski graphs. In [5] the authors propose an algorithm, which makes use of three automata and the fact that there are at most two internally vertex disjoint shortest paths between any two vertices, to determine all shortest paths in Sierpinski graphs. The authors of [3] proved that for any $n \geq 1$ and $l \geq 1$, the Sierpinski graph $S(K_n, l)$ has a unique 1-perfect code (or efficient dominating set) if l is even, and $S(K_n, l)$ has exactly n 1-perfect codes if l is odd. The Hamming dimension of a graph G was introduced in [6] as the largest dimension of a Hamming graph into which G embeds as an irredudant induced subgraph. That paper gives an upper bound for the Hamming dimension of the Sierpinski graphs $S(K_n, l)$ for $n \geq 3$. It also shows that the Hamming dimension of $S(K_3, l)$ grows as 3^{l-3}. The idea of almost-extreme vertex of $S(K_n, l)$ was introduced in [7] as a vertex that is either adjacent to an extreme vertex of $S(K_n, l)$ or is incident to an edge between two subgraphs of $S(K_n, l)$ isomorphic to $S(K_n, l-1)$. The authors of [7] deduced explicit formulas for the distance in $S(K_n, l)$ between an arbitrary vertex and an almost-extreme vertex. Also they gave a formula of the metric dimension of a Sierpinski graph, which was independently obtained by Parreau in her Ph.D. thesis. The eccentricity of an arbitrary vertex of Sierpinski graphs was studied in [8] where the main result gives an expression for the average eccentricity of $S(K_n, l)$. For a general background on Sierpinski graphs, the reader is invited to read the comprehensive survey [9] and references ther in.

This construction was generalized in [10] for any graph $G = (V, E)$, by defining the l-th generalized Sierpinski graph of G, denoted by $S(G, l)$, as the graph with vertex set V^l and edge set defined as follows, $\{u, v\}$ is an edge if and only if there exists $i \in \{1, 2, \ldots, l\}$ such that

(i) $u_j = v_j$, if $j < i$

(ii) $u_i \neq v_i$

(iii) $u_j = v_i$ and $v_j = u_i$ if $j > i$

In a compact form, the edge sets can be described as

$\{\{wu_iu_j^{d-1}, wu_ju_i^{d-1}\} : \{u_i, u_j\} \in E; d \in [t]; w \in V^{l-d}\}$

Figure 1 shows a graph G and the genenalized Sierpinski graph $S(G, 2)$, while Figure 2 shows the Sierpinski graph $S(G, 3)$

Notice that if $\{u,v\}$ is an edge of $S(G, l)$, then there is an edge $\{x,y\}$ of G and a word w such that $u = wxyyy....y$ and $v = wyxxx. x$. In general, $S(G, t)$ can be contructed recursively from G with the following process: $S(G, 1) = G$ and, for $t \geq 2$, we copy n times $S(G, l - 1)$ add the letter x at the begining of each label of the vertices belonging to the copy of $S(G, l - 1)$ corresponding to x. Then for every edge $\{x,y\}$ of G, add an edge between vertex $xyy....y$ and vertex $yxx. x$. See, for instance, Figure 2, vertices of the form $xx...x$ are called extreme vertices of $S(G, l)$. Notice that for any graph G of order n and any integer $t \geq 2$, $S(G, l)$ has n extreme vertices and, if x has degree $d(x)$ in G, then the extreme vertex $xx...x$ of $S(G, l)$ also has degree $d(x)$. Moreover the degrees of two vertices $yxx...x$ and $xyy\ y$, which connect two copies of $S(G, l - 1)$, are equal to $d(x) + 1$ and $d(y) + 1$, respectively.

For any $w \in V^{l-1}$ and $l \geq 2$, the subgraph (V_w) of $S(G, l)$, induced by V_w=$wx : x \in V$, is isomorphic to G. Notice that there exists only one vertex $u \in V_w$ of the form $w'xx. x$, where $w' \in V^r$ for some $r \leq t - 2$. We will say that $w'xx. x$ is the extreme vertex of (V_w), which is an extreme vertex in $S(G, l)$ whenever $r = 0$ By definition of $S(G, l)$

Suppose that W is an interconnection network (network for short). A path (cycle) in W is called a Hamiltonian path (Hamiltonian cycle) if it contains every node of W is called a Hamiltonian if there is a Hamitonian cycle in W, and it is called Hamiltonian-connected [15] if there is a Hamilto- nian path between every two distinct nodes of W. Some topologies, such as the hierarchical cubic network [16], are Hamiltonian-connected.

Since node faults and link faults may develop in a network, it is practically important to consider faulty networks. A network W is called $k-$ *node* (k-link) Hamiltonian if it remains Hamiltonian after removing any k nodes (links) [17]. If W has node (link) connectivity $k + 2$ and is $k -$ *node* (k-link) Hamiltonian, then it can tolerate a maximal number of node(link) faults while embedding a longest fault-free cycle. Some networks have been shown to be $k-$ *node* Hamiltonian and $k-$ *link* Hamiltonian. For example, the hierarchical cubic network with connectivity $n + 1$ is $n-$ 1-link Hamiltonian [18]. The $n-$ *dimensional* twisted cube [19] is $n-$ 2-node Hamiltonian and $n-$ 2-link Hamiltonian.

Note that an $n-$ *link* Hamiltonian graph cannot be guaranteed to be $n-$ *node* Hamiltonian. For example, the $n-$ *cube* is $n-$ 2-link Hamiltonian, but not $n-$ 2-node Hamiltonian [20]. In [21], the

WK-recursive network with connectivity $d-1$ was shown to be $d-3$-link Hamiltonian. It is not possible to reuse their approach of replacing link failures with node failures because a faulty node will cause $d-1$ links to fail and because their approach can handle at most $d-3$ faulty links.

2 Preliminaries

Each node of $S(W_k, l)$ is labelled as a $l-$ *digit* radix k number. Node $u_{l-1}u_{l-2}....u_1u_0$ is adjacent to 1. $u_{l-1}u_{l-2}....u_1v$, where $v \neq u_0$ and adjacent to 2. $u_{l-1}u_{l-2}....u_{j+1}v_1(v_0)^j$ if $u_j \neq u_{j-1}$ and $u_{j-1} = u_{j-2} = ...u_0$, where $v_1 = u_{j-1}$ $v_0 = u_j$ and $(b_0)^j$ denotes j consecutive v_0s. The links of 1 are called substituting links and are labelled 0. The link 2 is called a j flipping link and is labelled
j. For example, the 2-flipping link connects node 033 and node 300 in $S(W_5, 3)$. In addition, if
$u_{l-1} = u_{l-2} =$ u_0, then an open link labelled l is incident with $u_{l-1}u_{l-2}$ u_1u_0. The open link is reserved for further expansion; hence, its other end node is unspecified.

Note that $S(W_k, 1)$ is a k-node Wheel graph augmented with l open links. Each node of $S(W_k, l)$ is incident with $l-1$ substituting links and one flipping link (open link). The substituting links are those within basic building blocks, and the j-flipping links are those connecting two embedded $S(W_k, j)s$.

3 Hamiltonian-Connectedness

In this section, we show that $S(W_k, l)$ is Hamiltonian-connected, where $k \geq 4$ and $l \geq 1$

The basic idea is to use induction. Assume that we can prove that $S(W_k, l-1)$ is Hamiltonian- connected. We want to prove that $S(W_k, l)$ is also Hamiltonian-connected. That is, we want to construct a Hamiltonian path between two arbitrary distinct nodes $U = u_{l-1}u_{l-2}....u_1u_0$ and
$V = v_{l-1}v_{l-2}....v_1v_0$ in $S(W_k, l)$. When U and V are networks of level $l-1$,That is $u_{l-1} \neq v_{l-1}$, an $U-V$ Hamiltonian path for $S(W_k, l)$, can be constructed by connecting the $l-1$-flipping links and the Hamilton paths for each subnetwork of level $l-1$

When U and V are in the same subnetwork of level $l-1$. That is $u_{l-1} = v_{l-1}$, two $l-1$-frontiers in $u_{l-1}.S(W_k, l-1)$ are chosen. Then, two disjoint paths from U to one $l-1$-frontier and from V to another $l-1$-frontier, respectively, are constructed. These two disjoint paths must contain all the nodes in $u_{l-1}.S(W_k, l-1)$, the $l-1$-flipping links, and the Hamiltonian paths for the remaining subnetworks of level $l-1$. We can first prove a lemma for the existence of two such disjoint paths and then use the lemma to construct a $U-V$ -Hamiltonian path for $S(W_k, l)$. However, if $S(W_k, l-2)$

is Hamiltonian-connected, then constucting the two disjoint paths in $u_{l-1}.S(W_k, l-1)$ is Hence ,proving Hamiltonian-Connectedness and constructing two disjoint paths simultaneously will make the latter task shorter and easier(refer to the proof in Theorem 1). As a result, we combine them together in the following theorem:

Theorem 1. *Let $U = u_{l-1}u_{l-2}....u_1u_0$ and $V = v_{l-1}v_{l-2}....v_1v_0$ be two distinct nodes in $S(W_k, l)$, where $d \geq 4$*

1. *There is an $U - V$ Hamiltonian path for $S(W_k, l)$.*
2. *Given two distinct constants $a, b \in 0, 1, 2....k-1$ with $\{c, e\} \neq u_{l-1}, v_{l-1}$ there exists an $U - X$ path $V - Y$ path such that they are disjoint and contain all the nodes of $S(W_k, l)$ where $\{X, Y\} = \{(a)^l, (b)^l\}$ (Since $\{a, b\} \neq u_{l-1}, v_{l-1}\}$), we have $\{X, Y\} \neq \{U, V\}$, that is, it is possible that $U = X$ or $V = Y$ but not both. Note that if both $U = X$ and $V = Y$, then the $U - X$ path degenerates to a node U and the $V - Y$ path degenerates to a node V. As a result the $U - X$ path and the $V - Y$ path cannot contain all the nodes of $S(W_k, l)$*

Proof. We proceed by by induction on l. Clearly the theorem holds for $l = 1$. Assume it holds for $l = t \geq 1$. The situation in the case of $l = t + 1$ is discussed below. In the rest of the proof, we will use $\longrightarrow$ to denote a l-flipping link in $S(W_k, l+1)$. First we will prove part 1

Case 1: $u_l = v_l$, that is, U and V are not in the same subnetwork of level l. Let $a_0 = u_0$ and $\{b_0, b_1, ...b_{k-1}\} = \{0, 1, ...k-1\}$. Thus $b_0S(W_k, l), b_1S(W_k, l)$ $b_{k-1}S(W_k, l)$ denotes k subnetworks of level l. By assumption, there exists nodes in $b_iS(W_k, l)$ between two arbitrary distinct nodes in $b_iS(W_k, l)$ for each $i \in \{0, 1, 2...k-1\}$. Let $\overset{H}{\Rightarrow}$ denote this path. An $U - V$ Hamiltonian path for $S(W_k, l+1)$ constructed as follows (see Fig. 2):

$U \overset{H}{\Rightarrow} a_0(a_1)^l \longrightarrow a_1(a_0)^l \overset{H}{\Rightarrow} a_1(a_2)^l \longrightarrow .....\longrightarrow b_{k-2}(b_{k-3})^l \overset{H}{\Rightarrow} b_{k-2}(b_{k-1})^l \longrightarrow b_{k-1}(b_{k-2})^l \overset{H}{\Rightarrow} V$

Case 2: $u_l = v_l$

Let $a_0 = u_l$, $\{b_0, b_1, ...b_{k-1}\} = \{0, 1, 2...k-1\}$ and $\{b_1, b_{k-1}\} \neq \{u_{l-1}, v_{l-1}\}$. Thus it is impossible that $\{U, V\} = \{b_0(b_1)^l, b_0(b_{k-1})^l\}$. By assumption, there exist an $U - X'$ path and $V - Y'$ path such that they are disjoint and contain all the nodes of $b_0.S(W_k, l)$ where $\{X', Y'\} = \{b_0(b_1)^l, b_0(b_{k-1})^l\}$

Let $\overset{P1}{\Rightarrow}$ denotes the $U - X'$ path and let $\overset{P2}{\Rightarrow}$ denotes $V - Y'$. Also by assumption, there exists a Hamiltonian path for $b_i.S(W_k, l)$ between two arbitrary distinct nodes in $b_i.S(W_k, l)$ for each $i \in \{1, 2, ...l-1\}$. Let $\overset{H}{\Rightarrow}$ denotes this path without loss of generality assume that $X' = \{b_0(b_1)^l$ and $Y' = \{b_0(b_{k-1})$ An $\{U, V\}^l$ Hamiltonian path for $S(W_k, l)$ is constructed as follows (see fig3)

$U \overset{P1}{\Rightarrow} b_0(b_1)^l \longrightarrow b_1(b_0)^l \overset{H}{\Rightarrow} b_1(b_2)^l \longrightarrow .....\longrightarrow b_{k-2}(b_{k-2})^l \overset{H}{\Rightarrow} b_{k-1}(b_0)^l \longrightarrow b_0(b_{k-1})^l \overset{P2}{\Rightarrow} V$

□

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