JETIR.ORG ISSN: 2349-5162 | ESTD Year : 2014 | Monthly Issue
JOURNAL OF EMERGING TECHNOLOGIES AND INNOVATIVE RESEARCH (JETIR)
An International Scholarly Open Access, Peer-reviewed, Refereed Journal

AN EQUIVALENCES OF GEOMETRY OF TREES WITH MICRO-GEOMETRY

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ABSTRACT: There is a well known correspondence between infinite trees and ultra metric spaces that comes from considering the end space of the tree [1]. The correspondence is interpreted here as equivalence between two categories, one of which encodes geometry of trees at infinity and other encodes that micro-geometry of complete ultra metric spaces.

KEY WORDS: Ultra metric end spaces, Isometry, Trees, Similarity Equivalence.

0. INTRODUCTION

In this section we define the functor F from trees to ultra metric space. The functor takes a rooted tree to the end space of the tree, so we begin by defining the end space of a rooted R-trees and its natural metric. The end space functor F: $T \rightarrow U$ is indeed a functor. We shall prove that this functor is full and faithful.

1. FULL AND FAITHFULL FUNCTOR

The following concept is quite well known [2].

Definition 1.1: The end space of a rooted R-tree (T, v) is given by end

$(T, v) = \{f: [0, \alpha) \rightarrow T \mid f(0) = v$ and f is an isometric embedding$\}$

For $f, g \in$ end (T_1, v), we define

$$de(f,g) = \begin{Bmatrix} 0 & \text{if } f = g \\ \dfrac{1}{e^{t_0}} & \text{if } f \neq g \text{ and } t_0 = \sup\{t \geq 0 | f(t) = g(t)\} \end{Bmatrix}$$

where, $\{t \geq 0 | f(t) = g(t)\} = \begin{cases} [0, \alpha) & \text{if } f = g \\ [0, t_0) & \text{if } f \neq g \end{cases}$

Theorem 1.2 : If (T, v) is a rooted R-tree, then (end (T, v), de) is a complete ultra metric space of diameter $\leq$ 1.

Proof: To check the ultra metric inequality, let $f, g, h \in$ end (T, v) and show that

$de(f, q) \leq \max \{de(f, h), de(g, h)$

Without any loss of generality suppose that

$de(f,h) = e^{-t_1} \ngeq de(h,g) = e^{-t_2}$. Then $t_1 \leq t_2$,

$f = h$ on $[0, t_1]$ and $h = g$ on $[0, t_2]$. Thus $f = g$ on $[0, 1]$ and $de\ (f \cdot g) \leq e^{-t_1}$.

The statement about the diameter is obvious. To verify that (en (T, v), de) is complete, let $(f_1)_{i=1}^{\infty}$ be a Cauchy sequence in end (T, v). By passing to a subsequence we may assume that there is a non decreasing sequence of integers [3]

$1 \leq i_1, \leq i_2, \leq i_3, \leq \ldots\ldots\ldots$

that $f_i = f$; on $[0, 1]$, whenever $i, j \geq n$.

Define $f : [0, \infty) \to T$ by setting $f| [0, n] = f \mid [0, n]$ for each n.

Then f is well defined isometric embedding and $\lim_{i \to \alpha} f_i \to f$.

Theorem 1.3: Let (f, C_r, C_s) : (T, v) → (S, ω) be an isometry at infinity between geodesically complete rooted R-trees. Then there is an induced local similarity equivalence $f_* : \text{end}(T, v) \to \text{end}(S, \omega)$. Moreover if (g, C_T^l, C_S^l) is another such symmetry at infinity and $[f_*^1] = [g]$, then $f_* = g_*$.

Proof: In order to define f_*, let

$$\alpha : [0, \infty) \to T$$

be an element of end (T, v). Since C_r is a cut set, there exists a unique $t_p > 0$ such that $\alpha(t_0) \in C_T$. Moreover, $\alpha([t_0, \infty]) \leq T_{\alpha(t_0)}$.

Let $\hat{\alpha} : [0, f\hat{\alpha}(t_0)] \to S$ be the unique isometric embedding such that $\alpha(0) = \omega$ and $\hat{\alpha} : (f\hat{\alpha}(t_0)) \to f\alpha(t_0)$.

Define

$$f_*\alpha(t) = \begin{cases} \alpha(t), & \text{if } 0 < t \leq f\alpha(t_0) \\ f\alpha(t - f\alpha(t_0) + t_0), & \text{if } f\alpha(t_0) \leq t \end{cases}$$

Then clearly $f_*(\alpha) \in$ end (S, ω). To see that f is a local similarity equivalence, we will first show, given a as above, there exist $\in > 0$ and $\lambda > 0$ such that

$$f_*| : B(\alpha, \in) \to \beta(f(\alpha), \lambda \in$$

is a subjective λ similarity. Let $\in = e^{-t_0}$ and $\lambda = e^{t_0 - f\alpha|(t_0)}$.

If $\beta \in$ end (T, v) with $\alpha \neq \beta$ and $de(\alpha, \beta) < \in$, then

$t_\beta = \text{Sup}\{t \geq 0 \mid \alpha \mid (t) = \beta(t)\} > -\log \in = t_0$.

In particular, $de(\alpha, \beta) = e^{t_0}, \alpha(t_0) = \beta(t_0)$ and $f\alpha(t_0) = f\beta(t_0)$.

It follows that

$$f \cdot (b)(t) = \begin{cases} \hat{\alpha}(t) & \text{if } 0 \leq t \leq f\beta(t_0) \\ f\beta(t - f\beta(t_0)) + t_0, & \text{if } f\beta(t_0) < t, \end{cases}$$

$$= \begin{cases} \hat{\alpha}(t) & \text{if } 0 \leq t \leq f\beta(t_0) \\ f\alpha(t - f\alpha(t_0)) + t_0, & \text{if } f\alpha(t_0) \leq t \leq t\beta - t_0 + f\alpha(t_0) \\ f\beta(t - f\alpha(t_0)) + t_0, & \text{if if } t\beta - t_0 + f\alpha(t_0) \leq t, \end{cases}$$

Hence $\sup\{t \geq 0 | f \cdot \alpha(t) = f \cdot \beta(t)\} = t_\beta - t_0 + f\alpha(t_0)$

$$\text{and } de(f, \alpha, f \cdot \beta) = e^{-t_\beta + t_0 - f\alpha(t_0)}$$

$$= e^{t_0 - f_\alpha(t_0)} de(\alpha, \beta)$$

$$= \lambda de(\alpha, \beta)$$

To see that $f_*|B(\alpha, \in) \to B(f_*\alpha, \lambda \in)$ is subjective, let $\lambda \in B(f_* \alpha, \alpha \in)$

Then $d(\gamma, f_*\alpha) = \lambda \in = e^{-f\alpha(t_0)}$. It follow that

$$\gamma(f\alpha(t_0)) = f\alpha(t_0) \text{ and } \gamma([f\alpha(t_0), \infty]) \subseteq S_{f\alpha}(t_0).$$

Since $f| : T_\alpha(t_0) \to S_{f\alpha}(t_0)$ is an isometry,

we can define $\beta : [0, \infty] \to T$ by

$$\beta(t) = \begin{cases} \alpha(t), & \text{if } 0 \leq t \leq t_0 \\ \left(f|T_\alpha(t_0)t_0\right)^{-1} \gamma\left(t + f\alpha(t_0)\right) - t_0, & \text{if } t_0 \leq t, \end{cases}$$

Then $\beta \in$ end (T, v) and $\beta \in B(\alpha, \in)$ and $f_* \beta = \gamma$. Under the similar argument, we can show that: $f_* \cdot$ end $(T, v) \to$ end (S, ω) is surjective. Here are the detail.

If $\gamma \in$ end (S, ω), then there exists a unique $t_\gamma > 0$, such that $\gamma(t_\gamma) \in C_s$, and there exists a unique $c \in C_T$ such that $f(c) = \gamma(t_\gamma)$.

Let $\hat{\gamma} : [o, \|c\|] \to T$ be unique isometric embedding such that $\hat{\gamma}(o) = v$ and $\hat{\gamma}(\|c\|) = c$. Define $\beta : [0, \infty) \to T$ by

$$\beta(t) = \begin{cases} \hat{\gamma}(t) & \text{if } o \le t \le \|c\| \\ ((f|T_c)^{-1}\gamma(t + \|\gamma(t_c)\| - \|c\|), & \text{if } \|c\| \le t \end{cases}$$

To see that f_* is injective, suppose that $f_* \alpha = f_* \beta$ for some $\alpha, \beta \in$ end (T, v). Then there exists $t_1 > 0$ such that $\alpha(t_1, \infty) \cup = \beta(t_1, \infty)$ is a domain of f and $f\alpha([t_1, \infty] = f\beta([t_1, \infty))$.

Since f is a homeomorphism, it follows that $\alpha([t_1, \infty] = \beta([t_1, \infty))$ and hence $\alpha = \beta$. We know that if $c \in C_T$ and $x \in T_c$, then

$\|x\| - \|c\| = \|f(x)\| - \|f(c)\|$ [1, 2].

Hence f_* is independent of representation of $[f]$. The definition of $f_*(\alpha)$ will not change if another cut set $C_T{}'$ for (T, v) larger than C_T is used in place of C_T. For such a cut set, there exists a unique $t_1 > 0$ such that $\alpha(t_1) \in C_T{}'$. If follows that $t_1 \ge t_0$, $\alpha(t_1) \in Ta(t_0)$ and $f\alpha(t_1) = T_{fa}(t_0)$.

Let $\hat{\alpha}^1 : [0, \| f\alpha(t_1)\|] \to S$ be unique isometric embedding such that $\hat{\alpha}^1(0) = \omega$ and $\hat{\alpha}^1 \| f\alpha(t_1)\|] = f\alpha(t_1)$.

The map $f_*^1(\alpha) : [0, \infty) \to \infty$ is given by

$$f_*^1(\alpha)(t) = \begin{cases} \hat{\alpha}^1(t) & \text{if } 0 \le t \le \| f\alpha(t_1)\|, \\ f\alpha(t - \| f\alpha(t_1)\| + t_1) & \text{if } \| f\alpha(t_1)\| \le t_1. \end{cases}$$

would be defined if $C_T{}'$ were used in place of C_T. However, since S is an R-tree and $\alpha_1(\| f\alpha(t_0)\|) = f\alpha(t_0)$, if follows that $\alpha_1[0, \| f\alpha(t_0)\|] = \alpha$. Also we have

$$t_1 - t_0 = \|\alpha(t_1)\| - \|\alpha(t_0)\| = \| f\alpha(t_1)\| - \| f\alpha(t_0)\|$$

and, hence $t_1 - \| f\alpha(t_1)\| = t_0 - \| f\alpha(t_0)\|$.

It follows that $f_1^*(\alpha) = f_*(\alpha)$ and $f_1^* = f_*$.

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